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Research Article

On Isometrically Equivalent Norms in Banach Spaces

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Abstract

Many studies on equivalent norms are playing an increasingly important role in the theory of Banach spaces. Also in consideration has been the characterization of operator ideals. In this paper we develop isometrically equivalent norms in a Banach space. The objectives of the study are to show that there exists Banach spaces which are isometrically isomorphic; develop an unconditional basis in a Banach space and renorm an equivalent norm under which a Banach space will become a Banach lattice and still remain a banach space. Authors showed that there exists Banach spaces which are isometric and isomorphic. Authors further showed that for a normal Banach space which is absolutely continuous, then there exists an unconditional basis in a Banach space.

Keywords: Banach space; Banach lattice; Unconditional basis; Isomorphic.

Introduction

A crucial point in that analysis is the concept of level sequence, which is introduced and discussed. As an application, the best constant is derived in the triangle inequality for such class [24]. The notion of hermitian operators in Hilbert spaces has been extended to Banach spaces by Lumer and Vidar. Recently, Berkson has shown that a scalar type operator S in Banach space X can be decomposed into S=R+iJ where (i) R and J commute and (ii) $R^m J^n (m, n = 0, 1, 2)$ are hermitian in some equivalent norm on X. The converse is also valid if the Banach space is reflexive. Thus we see that the scalar type operators in a Banach space play a role analogous to the normal operators in Hilbert spaces.[26]. In unitary operators in Banach spaces, the well known Hilbert space no- tion of unitary operators was suitably extended to operators in Banach spaces and a polar decomposition was obtained for a scalar type oper- ator. It was further shown that this polar decomposition is unique and characterizes scalar type operators in reflexive Banach spaces. Finally, an extension of stone's theorem on one parameter group of unitary operators in Hilbert spaces was obtained (under suitable conditions) for reflexive Banach spaces [26].

It was proved that every Banach space containing a complemented copy of C_O has an antiproximinal body for a suitable norm, if in addition, the space is separable, there is a pair of antiproximinal norms. In particular, in separable polyhedral space X, the set of all (equivalent) norms on X having an isomorphic antiproximinal norm is dense. In contrast , it was shown that there are no antiproximinal norms in Banach spaces with the convex point of continuity property ($CPCP$). Other questions related to the existence of antiproximinal bodies were also discussed [16]. Two geometric concepts of a Banach space propertyα and β, were de- fined which generalize in a certain way the geometric situation l and c_O. These properties have been used by J. Lindestrauss and J Partington in the study of norm $(3 + \varepsilon)$ equivalently be renormed to have property β. It was shown that many Banach spaces (eg, every WCG space), may $(3 + \varepsilon)$-equivalently be renormed to have property α. However an example due to 'S. Shelah' shows that not every Banach space is isomorphic to a Banach space with property α [27].

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For any Banach space X there is a norm $\| \cdot \|$ on X, equivalent to the original one, such that $(X, \|\cdot\|)$ has only trivial isometries. For any group G there is a Banach space X such that the group of isometries of X is isomorphic to $G \times -1, +1$. For any countable group G there is a norm $\| \cdot \|_{tt}$ on $C([0, 1])$ equivalent to the original one such that the group of isometries of $(C([0, 1]), \| \cdot \|_{tt}$ is isomorphic to $G \times -1, +1$ [21]. Many researchers have devoted their time to persue studies in the theory of Banach spaces. Due to certain conditional properties of basis in Banach space in relation to isomorphism, research in the area of equiv- alent norms remain an interesting area of study. For any Banach space X there is a Banach space Y with $X \subseteq Y$ such that Y has only trivial isometries [21]. It was also proven that for any group G there is a compact connected Housdorff S such that the group of all isometries of compact Housdorff space and $G \times -1, 1$ are isomorphic. For any group G there is a metric connected space such that the group of all isometries of Lip(S) is isomorphic to the product of G the multiplicative group of the unit circle [21]. The author considers a norm which is equivalent to the original one but uses mainly trivial isometries. In all cases the author did not use the unconditional basis in the proof. In this research we shall use the unconditional basis and renorm an isometrically equivalent norm under which a Banach space will become a Banach lattice and still remain Banach.

Research methodology

In this research we show that for a normal Banach space which is absolutely contin- uous, then there exists an unconditional basis in λ. We come up with a sequence which shall be assumed to have a conditional basis and we shall choose an infinite sequence. We will show that if λ is a Banach space , then there exists λ* and λ^j which are isometric and isomorphic. Authors came up with a schauder basis and apply Banach-Steinhaus theorem and the sequence convergence. We further show that the norms are equivalent under which λ will become a Banach lattice which we shall renorm. We apply the principle of uniform boundedness and prove that the norms are equiva- lent. By applying the duality theory and considering the dual pair with a bilinear form, we will have developed equivalent norms in a Banach space. Generally , let E be a Banach space with a basis (e_n). Clearly we are free to multiply each (e_n) by a non zero scalar without changing the basis property . Hence we may assume that (and we will henceforth do so). (e_n) is a normalised basis, that is $\|e_n\| = 1$ for each n.

Results and discussion

In this section, authors presented the results of their work.

Proposition 3.1

Let E be a banach space with a basis (e_n) and let $\| \cdot \|_0$ be as defined above. Then $\| \cdot \|$ and $\| \cdot \|_0$ are equivalent norms on E.

Proof. Let F be the normed space with each E together with the norm $\| \cdot \|_0$.(It is easily checked that F is indeed a normed space). Let $i : F \to E$ be the formal inclusion map. Then i is a norm decreasing, and is a bijection. Suppose that F is a Banach space, that is F is complete. Then the open mapping theorem implies that i has a continous inverse, which implies that $\| \cdot \|_0$ is equivalent to$\| \cdot \|$. This completes the proof.

Proposition 3.2

Let X be a banach space and Y be a normed space. Suppose the sequence $T_n \subset B(X, Y)$ has the property that for every $x \in X$, the sequence $T_n(x) \subset Y$ is bounded. Then the sequence $\| T_n(x) \| \subset$ R of norms is bounded.

Proof. **W**e say if X is Banach, then point wise boundedness implies uniform boundedness. Let X be a banach space, Y a normed space, and $T_n \subset B(X, Y)$ be a point wise bounded sequence of operators. Let us define the sets $A_k := x \in X \| T_n(x) \| \leq k$. It is easy to see that these are closed sets, and the fact that T_n is a point wise bounded sequence implies that X closed sets contains an open ball say $B_0 := B(x_0, r) \subset A_{k0}$. We therefore have that $\| T_n(x) \| \leq k_0$ for any $x \in B_0$ and $n \in N$ (This is all we need Baire's category theorem for in this proof). Now let $x \in X$, of course, there is some $z \in B_0$ such that $z - x_0$ ”points” in the direction of x, meaning that $x = R(z - x_0)$ for some $R > 0$. Infact we may choose Z with the added restriction that it be a distance. Proving that the sequence is uniformly bounded [3]. The rest is clear.

Theorem 3.3

A subclass Ω of L is called an ideal of operator if the so called components $\Omega(X, Y) := \Omega \cap L(X, Y) = S \in L(X, Y) : S \in \Omega$ of Ω satisfy the following three conditions:

1. $a \otimes y \in (X, Y) \forall a \in X^J$ and $y \in Y$. Here $a \otimes y : X \to Y$ denotes the mapping $(a \otimes y)(X) = (a, X)Y$,
2. $S + T \in \Omega(X, Y)$ for all $S, T \in \Omega(X, Y)$
3. $R \cdot S \cdot T \in \Omega(X_0, Y_0)$ for all X_0, Y_0 in banach space and $T \in L(X_0, X)$, $S \in \Omega(X, Y)$ and $R \in (Y, Y_0)$.

Proof. Obviously the above definition of an ideal of operators can be extended to define , in a similar way ideals of mappings on the families of banach and normed spaces. The restrictions Ω*/banach* spaces and Ω*/normed* spaces of an ideal Ω of mappings on banach to the families of banach spaces and all normed spaces respectively and again will be ideals on these families. If I is an operator ideal on banach space and Ω is an ideal on mappings on banach space such that $\Omega/banach = I$. Then Ω is called an extension of I to banach space. Owing to [19] there are always a minimum and maximum extention of a given operator ideal Ω on banach space to banach space denoted by Ω^{inf} and Ω^{sup} respectively. Hence for any extension I of Ω hold $\Omega^{inf} \subseteq I \subseteq \Omega^{sup}$. The extension Ω^{if} is defined in the following way. Consider banach space E and F. Then $T \in \Omega^{inf}(E, F)$ if and only if, there exists a factorization $T = Q \cdot S \cdot P$ with $P \in L(E, X)$, $S \in \Omega(X, Y)$, $Q \in L(Y, F)$ and X and Y banach spaces. [2]. The family P of precompact linear mappings on banach space, in [5] the ideal QS of quasi- Schwartz mappings on locally convex spaces was introduced. Owing to [19] the ideal QS is exactly the minimal extension to banach space of P/norms. An ideal Ω of mappings on banach space is called injective if for each continuous open linear injection $J \in L(F, G)$. It follows from $S \in L(E, F)$ and $J \cdot S \in \Omega(E, G)$ that $S \in \Omega(E, F)$. It is easy to see that any given ideal Ω of mappings on banach space is contained in an injective Ω^i on banach space , which is the smallest among all injective ideals of mappings in banach space containing Ω. The ideal Ω^i is simply the intersection of all injective ideals containing Ω and is called the injective hull of Ω. In the same way the injectivity (or else the injective hull) is defined for operator ideals on banach space. The following characterization for Ω^i on banach space is defined. An operator $T \in L(X, Y)$ is in Ω^i if and only if there exists a banach space Z such that Y is isometrically isomorphic to the subspace of Z such that $J \cdot T \in\subseteq \Omega(X, Z)$, where $J : Y \to Z$ is the isometry onto a subspace of Z. since Y^{∞} has the extension property for all y in banach space.

Corollary 3.4.

Every norm in a Banch space is isometrically equivalent.

Proof. Since T is continuous, it is seen from (2) that $y = T_x$. Thus it has been shown that an arbitrary ball X_r about the origin X maps onto a set TX_r which contains a ball $Y_s = Y_t0$ about the origin Y. A basis $\{X_n : n \in N\}$ in a banach space X is said to be shrinking if the associated sequence of coefficient functional $\{f_n : n \in N\}$ is a basis in X^J. Clearly every shrinking basis is a schauder basis. The natural basis for C_0 and Ap ($1 < p < \infty$), for instance are shrinking . we refer to[11] for more examples of banach spaces with more shrinking basis. Equivalent to our definition is the one given in [11]. For 0-shrinking (or shrinking basis). A basis $\{X_n : n \in N\}$ for a banach space X is called K-shrinking if, for the associated sequence of coefficient functional $\{f_n : n \in N\} \subset X^J$. We have codim $X^J[f_n] = K$ where $[f_n]$ denotes the subspace generated by $\{f_n : n \in N\}$ Let λ be a normal banach space which is absolutely continuous. In order to apply the duality theory, we sometimes find it necessary to assume that $\lambda^J(= \lambda^*)$ is absolutely continuous. We therefore define a banach space to be a normal banach space which is absolutely continuous

Conclusions

In the present work, authors have observed that there exists a normal banach space which is absolutely continuous and that $\{e_n : n \in N\}$, is a schauder basis for a normal banach space. Also that there exists a topological isomorphism in the same banach space.

Authors have observed that if λ is a normal banach space which is absolutely continuous , then λ^* and λ^J are isometrically isomorphic banach spaces. We have also established that if λ is a normal banach space which is absolutely continuous then $\{e_n : n \in N\}$ is an unconditional basis in λ . We have further observed that if λ is a normal banach space which is absolutely continuous, then λ can be renormed by an equivalent norm , under which λ will become a banach lattice and still remain a banach space which is absolutely. Thus in chapter four we have shown that there exists a banach space which is isometrically isomorphic, in which we have developed an unconditional basis in the banach space and finally we have renormed an equivalent norm under which a banach space becomes a banach lattice and still remain a banach space.

Conflicts of interest

Authors declare no conflict of interest.

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