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Research Article

Numerical radii inequalities for derivations induced by orthogonal projections

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Abstract

Let H^n be a finite dimensional Hilbert space and $\delta_{P,Q}$ be a generalized derivation induced by the orthogonal projections P and Q. In this study, authors have approximated the norm of $\delta_{P,Q}$ and showed that $\delta_{P,Q}$ is bounded and is utmost equal to the sum of norms of P and Q. Authors have also considered the linearity and inequalities of in the context of tensor products. Finally, authors have determined the Hilbert-Schmidt and completely bounded norm inequalities for $\delta_{P,Q}$.

Keywords: Hilbert space; Orthogonal projections; Hilbert-Schmidt; Inequalities.

Introduction

Indeed, it was noted in [4] that for any operator $T \in B(H)$ and norm ideal τ in $B(H)$, $diam(W(T)) \leq \| \delta_T | \tau \|$ where *'diam'* is the diameter. Furthermore, it was shown that if $T \in B(H)$ is S-universal, and τ a norm ideal in $B(H)$, then $diam(W(T)) \leq \| \delta_T | \tau \|$.

In [17], Rosenblum determined the spectrum of an inner derivation, $\delta_T = TP - PT$. Kadison, Lance and Ringrose [16] investigated derivations δ_T acting on a general C^*-algebra and which are induced by Hermitian operators. Stampfli [20] studied a derivation δ_T acting on an irreducible C^*-algebra $B(H)$ for all bounded linear operators on a Hilbert space H. The geometry of the spectrum of a normal operator T was used in [16] to show that the norm of a derivation is given by $\| \delta_T \| = inf\{2 \| T - \lambda \| : \lambda \in C\}$ using the geometry of the spectrum of normal operator T. Stampfli [20] raised the question on the ability to compute the norm of a derivation on an arbitrary C^*-algebra. Kaplansky [8] later used the density theorem to prove that the extension of derivations of a C^*- algebra to its weak-closure in $B(H)$ [8] is achieved without increasing norm.

Gajendragadkar [8] computed the norm of a derivation on a von Neumann algebra. Specifically it was shown that if φ is a von Neumann algebra of operators acting on a separable Hilbert space H and $T \in \varphi$ and δ_T is the derivation induced by T, then $\| \delta_T | \varphi \| = 2\, inf\{T - Z : Z \in C\}$ where C is the center of φ [6]. Given an algebra of bounded linear endomorphisms $\mathcal{L}(\mathcal{X})$ for a real or complex vector space $\mathcal{X}$, it was shown that for each element $T \in L(\mathcal{X})$, an operator $\delta_T(A) = TA - AT$ is defined on $\mathcal{L}(\mathcal{X})$ and $\| \delta_T \| \leq 2\, inf_\lambda \| T + \lambda I \|$. Furthermore if $\mathcal{X}$ is a complex Hilbert space then the norm equality holds [7]. Johnson [7] used a method which applies to a large class of uniformly convex spaces to show that this norm formula does not apply for ℓ^p and $L^p(0,1), 1 < p < \infty,\ p \neq 2$. For L^1 spaces, the formula was proved to be true in the real case but not in the complex case when the space has three or more dimensions.

The derivation constant $K(\mathcal{A})$ has been studied for unital non-commutative C^*-algebra $\mathcal{A}$ [2]. Archbold [2] studied $K(M(\mathcal{A}))$ for the multiplier $M(\mathcal{A})$ for a non-unital C^*-algebra $\mathcal{A}$ and obtained two results; that $K(M(\mathcal{A})) = 1$ if $\mathcal{A} = C^*(G)$ for a number of locally compact group G and $K(M(\mathcal{A})) = \frac{1}{2}$ if G is (non-abelian) amenable group. Salah [18] showed that in both finite and infinite dimensional vector spaces, the norm of a generalized derivation is given by $\| \delta_{A,B} \| = \| A \| + \| B \|$ for a pair $A, B \in B(H)$. Okelo in [15] and [14], showed the necessary

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and sufficient conditions for a derivation δ_T to be norm-attainable.

Several other results exists on the inequalities of derivations and commutators on C^*-algebras. For instance Kittaneh [9] used a polar decomposition $T = UP$ of a complex matrix T and unitarily invariant norm $|||.|||$ to prove the inequality $||| \, |UP - PU|^2 \, ||| \leq \| \, |T^*T - TT^*| \, \| \leq \| UP + PU \| \, \| UP - PU \|$.

Williams [22] proved that if a commutator $TX - XA = \alpha I$ is such that A is normal, then the norm relation $\| I - (TX - XT) \| \geq \| I \|$ holds. Anderson [1], generalized Williams inequality and proved that $\| P - (TX - XT) \| \geq \| P \|$. Later, Salah [18] proved that if T and P are normal operators, then $I - (TX - XP) \geq \| I \|$. The norms of derivations implemented by S-universal operators have been shown to be less than or equal to half the sum of inner derivations implemented by each operator in [5] and in particular was proved that, $\| \delta_{T,P} \| \leq \frac{1}{2} (\| \delta_{T-\lambda} \| + \| \delta_{P-\lambda} \|)$ and $\| \delta_{T-\lambda, P-\lambda} \| \leq \frac{1}{2} (\| \delta_{T-\lambda} \| + \| \delta_{P-\lambda} \|)$. Using unitaries and non-orthogonal projections, Bhatiah and Kittaneh [3] determined max-norms and numerical radii inequalities for commutators.

Some authors have used the concept of classical numerical range to study different classes of matrices of operators. For instance, many alternative formulations of (p, q)-numerical range $W_{p,q}(A) = \{E_p((UAU^*)[Q])$ for a unitary U where $1 \leq p \leq q \leq n$ for an $n \times n$ complex matrix X, with $q \times q$ leading principle submatrix $X[q]$ and the p^{th} elementary symmetric functions of the eigen values of $X[q]$ [11]. Chi-Kwong Li [10] extended the results of these formulations to the generalized cases, gave alternative proofs for some of them like convexity and even derived a formula for (p, q)-numerical radius of a derivation as $r_{p,q}(T) = max\{|\mu| : \mu \in W_{p,q}(T)\}$.

Mohammad [13] applied positive operators in the proof of a similar result. Orthogonal projections being bounded operators, have extensive uses on implementation of derivations and construction of underlying algebras of the derivations. Vasilevski [21] studied the applications of C^*-algebras constructed by orthogonal projections to Naimark's dilation theorem. Spivack [19] used orthogonal projections to induce a derivation on von Neumann algebras. In [12] Matej used mutually orthogonal projections acting on a C^*-algebra to prove that any local derivation is a derivation.

Basic definitions

Definition 2.1.

Let $T \in B(H)$. For an identity operator I, the set of scalars λ, for which $det(T - \lambda I)x = 0$ and $x \in H$, are called the eigenvalues (singular values) of T. The set of vectors $x \in H$ for which $det(T - \lambda I)x = 0$, are called eigenvectors. The set of eigenvalues λ for which $det(T - \lambda I)x = 0$, is called the spectrum of T and is denoted by $\sigma(T) = \{\lambda \in C : det(T - \lambda I)x = 0\}$

Definition 2.2.

Let $\varphi : B(H) \to B(H)$ be a linear map between operator spaces. For any given $n = 1$, we denote by $\varphi_n : M_n(B(H)) \to M_n(B(H))$ the linear map defined by $\varphi_n([a_{ij}]) = [\varphi(a_{ij})]$. A map $\varphi : B(H) \to B(H)$ is called completely bounded (in short c.b.) if $sup_{n \geq 1}\{\|\varphi_n\|, \varphi_n : M_n(B(H)) \to M_n(B(H))\} < \infty$.

Definition 2.3.

If $T \in B(H)$, then the operator $T^* : H \longmapsto H$ defined by $\langle Tx, y \rangle = \langle x, T^*y \rangle \; x, \; y \in H$ is called the adjoint of T.

Definition 2.4.

An operator $T \in B(H)$ is said to be self-adjoint if $T^* = T$ and if T is linear on a linear subspace M of a Hilbert space $\mathcal{H}$ into $\mathcal{M}$ the it is said to be Hermitian if in addition, $\langle Tx, y \rangle = \langle x, Ty \rangle, \; \forall \, x, \; y \in \mathcal{M}$

Results and discussion

In this section, we give some numerical radius inequalities for δ_{PQ} and also consider in equalities which involve both the numerical radius $w(\delta_{PQ})$ and its norm $\| \delta_{PQ} \|$. We will also discuss power inequalities the numerical radius inequalities.

Proposition 3.0.0.

Let $P, Q \in P_0(H)$ and $\delta_{P,Q}(X)$ be positive, then $w(\delta_{P,Q}(X)) \leq w(P^2X + XQ^2) - 2|\langle PXx, XQ\rangle|$

Proof. For some $x \in (H)^n$

$$\langle (P-Q)x, x\rangle \leq \langle (P-Q)^2x, x\rangle$$
$$= \langle (P-Q)x, (P-Q)x\rangle$$
$$= \langle Px, Px\rangle - 2|\langle Px, Qx\rangle| + \langle Qx, Qx\rangle$$
$$= \langle PP^*x, x\rangle - 2|\langle Px, Qx\rangle| + \langle QQ^*x, x\rangle$$
$$= \langle P^2x, x\rangle - 2|\langle Px, Qx\rangle| + \langle Q^2x, x\rangle$$
$$= \langle (P^2 + Q^2)x, x\rangle - 2|\langle Px, Qx\rangle|$$
$$\langle (P-Q)x, x\rangle \leq \langle (P2 + Q2)x, x\rangle - 2|\langle Px, Qx\rangle|.$$

This implies that

$$|\langle (P-Q)x, x\rangle| \leq |\langle (P^2+Q^2)x, x\rangle| - 2|\langle Px, Qx\rangle|.$$

So for the positive $X \in B(H)$, then

$$|\langle (PX - XQ)x, x\rangle| \leq |\langle (P^2X + XQ^2)x, x\rangle| - 2|\langle PXx, XQx\rangle|$$

and therefore

$$w(\delta_{P,Q}(X)) \leq w(P^2X + XQ^2) - 2|\langle PXx, XQ\rangle|$$

Proposition 3.0.1.

Let $P, Q \in P_0(H)$ and $X \in B(H)$ be positive operators, then

$$\left|\left\langle \begin{bmatrix} X^{\frac{1}{2}}P^{\frac{1}{2}}X^{\frac{1}{2}} & X^{\frac{1}{2}}P^{\frac{1}{2}}Q^{\frac{1}{2}} \\ Q^{\frac{1}{2}}X^{\frac{1}{2}}P^{\frac{1}{2}} & Q^{\frac{1}{2}}X^{\frac{1}{2}}Q^{\frac{1}{2}} \end{bmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix}, \begin{pmatrix} x_n \\ y_n \end{pmatrix}\right\rangle\right| \leq \left\|X^{\frac{1}{2}}(P-Q)X^{\frac{1}{2}}\right\|^2$$

, for a sequence of unit vectors x_n and y_n $n \geq 1$.

Proof. Suppose that P, Q and X are positive.

Then they have positive square roots $P^{\frac{1}{2}}$, $Q^{\frac{1}{2}}$ and $X^{\frac{1}{2}}$ such that

$$\| X^{\frac{1}{2}}(P-Q)X^{\frac{1}{2}} \|^2 \geq \| X(P-Q) \| = \| X^{\frac{1}{2}}(P-Q)X^{\frac{1}{2}} \|$$
$$= \left\|\begin{bmatrix} P^{\frac{1}{2}}X^{\frac{1}{2}} & -X^{\frac{1}{2}}Q^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} P^{\frac{1}{2}}X^{\frac{1}{2}} & 0 \\ X^{\frac{1}{2}}Q^{\frac{1}{2}} & 0 \end{bmatrix}\right\| = \left\|\begin{bmatrix} P^{\frac{1}{2}}X^{\frac{1}{2}} & 0 \\ X^{\frac{1}{2}}Q^{\frac{1}{2}} & 0 \end{bmatrix}\begin{bmatrix} P^{\frac{1}{2}}X^{\frac{1}{2}} & -X^{\frac{1}{2}}Q^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix}\right\| =$$
$$\left\|\begin{bmatrix} X^{\frac{1}{2}}P^{\frac{1}{2}}X^{\frac{1}{2}} & X^{\frac{1}{2}}P^{\frac{1}{2}}Q^{\frac{1}{2}} \\ Q^{\frac{1}{2}}X^{\frac{1}{2}}P^{\frac{1}{2}} & Q^{\frac{1}{2}}X^{\frac{1}{2}}Q^{\frac{1}{2}} \end{bmatrix}\right\|$$

let x_n, and y_n $(n \geq 0)$ be a sequence of unit vectors in H_n such that $\langle P^{\frac{1}{2}}XQ^{\frac{1}{2}}y_n, x_n\rangle \to \| P^{\frac{1}{2}}XQ^{\frac{1}{2}} \|$

then $$\left|\left\langle \begin{bmatrix} X^{\frac{1}{2}}P^{\frac{1}{2}}X^{\frac{1}{2}} & X^{\frac{1}{2}}P^{\frac{1}{2}}Q^{\frac{1}{2}} \\ Q^{\frac{1}{2}}X^{\frac{1}{2}}P^{\frac{1}{2}} & Q^{\frac{1}{2}}X^{\frac{1}{2}}Q^{\frac{1}{2}} \end{bmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix}, \begin{pmatrix} x_n \\ y_n \end{pmatrix}\right\rangle\right| \leq \left\|X^{\frac{1}{2}}(P-Q)X^{\frac{1}{2}}\right\|^2$$

Theorem 3.0.2.

Let $P, Q \in P_0(H)$, $X \in B(H)$, $0 < a < 1$ and $n \geq 1$. Then

$$2w^n\delta_{P,Q}(X) \leq \|\delta_{P,Q}(X)\|^{2an} + \|\delta_{P,Q}(X)\|^{2n}$$

Proof.

For each unit vector $x \in H$ we have

$$\|\langle (PX - XQ)x, x\rangle\| \leq \langle \| PX - XQ \|^{2a} x, x\rangle^{\frac{1}{2}} \langle \|(PX - XQ)^*\|^{2(1-a)} x, x\rangle^{\frac{1}{2}}$$
$$\leq \left(\frac{\langle \|PX - XQ\|^{2a} x,x\rangle^n + \langle \|(PX - XQ)^*\|^{2(1-a)} x,x\rangle}{2}\right)^{\frac{1}{n}}$$
$$\leq \left(\frac{\langle \|PX - XQ\|^{2an} x,x\rangle^n + \langle \|(PX - XQ)^*\|^{2(1-a)n} x,x\rangle}{2}\right)^{\frac{1}{n}}$$

so that

$$\|\langle (PX - XQ)x, x\rangle\|^n \leq \frac{1}{2}\left\langle \| PX - XQ \|^{2an} + \|(PX - XQ)\|^{2(1-a)n} x, x\right\rangle$$
$$= \frac{1}{2}\langle \| PX - XQ \|^{4an} + \| (PX - XQ) \|^2 x, x\rangle.$$

Therefore,

$$w^n = sup\{\|\langle (PX - XQ)x, x\rangle\|^n : x \in H \| x \| = 1\}$$
$$\leq \frac{1}{2}\langle \| P - Q \|^{4an} + \| (PX - XQ) \|^2 x, x\rangle : \}$$
$$= \frac{1}{2} \| PX - XQ \|^{2an} + \| (PX - XQ) \|^2.$$

Then for an arbitrary $X \in B(H)$, then

$$2w^n(PX - XQ) \leq \|\| PX - XQ \|^{2an} + \| (PX - XQ)^* \|^2 \|.$$

Theorem 3.0.3.

Let $P, Q \in P_0(H)$, $0 < a < 1$ and $n \geq 1$. Then

$$w^{2n}(PX - XQ) \leq a \| PX - XQ \|^{2n} + (1 - a) \| (PX - XQ)^* \|^{2n}$$

Proof.

For any unit vector $x \in H$, we have

$$\| \langle (PX - XQ)x, x\rangle \|^2 \leq \langle \| (PX - XQ) \|^{2a} x, x\rangle \langle \| (PX - XQ)^* \|^{2(1-a)} x, x\rangle$$
$$\leq \langle \| (PX - XQ) \| x, x\rangle^a \langle \| (PX - XQ) \| x, x\rangle^{(1-a)}$$
$$\leq (a\langle \| (PX - XQ) \| x, x\rangle^a + (1-a)\langle \| (PX - XQ) \| x, x\rangle^n)^{\frac{1}{n}}$$

$\leq \left(a \left\langle \|(PX - XQ)\|^n \ x,x \right\rangle^a + (1 - a)\langle \| (P - Q) \|^n \ x,x\rangle^n \right)^{\frac{1}{n}}$

$= a(2\langle \| (PX - XQ) \|^n \ x,x\rangle^a + a\langle \| (PX - XQ) \|^n \ x,x\rangle^n)^{\frac{1}{n}},$

so that

$\| \langle (PX - XQ)x,x\rangle \|^{2n} \leq 2(a\langle \| (PX - XQ) \|^n \ x,x\rangle^a + a\langle \| (PX - XQ) \|^n \ x,x\rangle^n).$

Therefore on post and premultiplication of the operator $P - Q$ by $X \in B(H)$, we obtain

$$\begin{aligned} w^{2r}(PX - XQ) &= sup\{\| \langle (PX - XQ)x,x\rangle \|^{2n} : x \in H, \| x \| = 1\} \\ &\leq sup\{2(a\langle \| PX - XQ \|^n \ x,x\rangle^a + a\langle \| PX - XQ \|^n \ x,x\rangle^n) : x \in H, \| x \| = 1\} \\ &= 2a \| PX - XQ \|^a + \| PX - XQ \|^n \end{aligned}$$

Lemma 3.0.4.

Let $P, Q \in P_0(H)$ be non-zero operators on a complex finite dimensional Hilbert space H. If $\lambda_1, \lambda_2 \in C\backslash\{0\}$ and $p_1, p_2 > 0$ are such that $\| P - \lambda_1 I \| \leq p_1$ and $\| Q - \lambda_2 I \| \leq p_2$ for an identity matrix $I : H \rightarrow H$, then

$\| PX - XQ \| \leq 2w(PX - XQ) + \frac{|p_1 - p_2|}{2|\lambda_2 - \lambda_1|}$.

Proof.

From the condition $\| P - \lambda_1 I \| \leq p_1$, we get that $\| (P - \lambda_1)x \| \leq \| P - \lambda_1 I \| \leq p_1$ for an arbitrary vector $x \in H^n$, $\| x \| = 1$. Similarly $\| (Q - \lambda_2)x \| \leq \| Q - \lambda_2 I \| \leq p_2$.

Therefore,

$\| Px - xQ + (\lambda_2 - \lambda_1)x \| \leq \| P - Q + (\lambda_2 - \lambda_1)I \| \leq p_1 - p_2.$

Thus

$\| (P - Q)(x) \|^2 + |\lambda_2 - \lambda_1|^2 \leq 2Re[(\lambda_2 - \lambda_1)\langle (P - Q)x,x\rangle] + p_1 - p_2$

$\leq 2|\lambda_2 - \lambda_1||\langle (P - Q)x,x\rangle| + p_1 - p_2.$

By obtaining the supremum over all $x \in H$, $\| x \| = 1$ in the inequality, we get that

$\| P - Q \|^2 + |\lambda_2 - \lambda_1|^2 \leq 2w(P - Q)|\lambda_2 - \lambda_1| + p_1 - p_2.$

By applying

$2 \| P - Q \| |\lambda_2 - \lambda_1| \leq \| P - Q \|^2 + |p_1 - p_2|^2,$

into the simplified inequality, gives the inequality,

$2 \| P - Q \| |\lambda_2 - \lambda_1| \leq 2w(P - Q)|\lambda_1 - \lambda_2| + p_1 - p_2.$

Post and premultiplication of $(P - Q)$ by $X \in B(H)$ gives the desired inequality.

Theorem 3.0.5.

Let $P, Q \in P_0(H)$ be non-zero operators on a complex finite dimensional Hilbert space H and $\lambda_1, \lambda_2 \in C\backslash\{0\}$, and $p_1, p_2 > 0$ with $\lambda_1 > p_1$, $\lambda_2 > p_2$. If $\| P - \lambda_1 I \| \leq p_1$ and $\| Q - \lambda_2 I \| \leq p_2$, then

$\| PX - XQ \|^2 - \frac{|p_1 - p_2|^2}{|\lambda_2 - \lambda_1|^2} \leq w^2(PX - XQ).$

Proof.

The inequalities

$\| P \|^2 + |\lambda_1|^2 - p_1^2 \leq 2|\lambda_1| w(P)$ and $\| Q \|^2 + |\lambda_2|^2 - p_2^2 \leq 2|\lambda_2| w(Q)$ hold and imply that

$\| P - Q \|^2 + |\lambda_1 - \lambda_2|^2 - |p_1 - p_2|^2 \leq |\lambda_1 - \lambda_2| w(P - Q)$

, which further simplifies to $\| P - Q \|^2 + |\lambda|^2 \leq 2|\lambda| w(P - Q)$ for $\lambda = \lambda_1 - \lambda_2$ and $p = p_1 - p_2$. Now since $|\lambda_1| > p_1$ and $|\lambda_2| > p_2$, then $\sqrt{|\lambda|^2 - p^2}$ is strictly positive and therefore, dividing the simplified inequality by $\sqrt{|\lambda|^2 - p^2}$ is meaningful and yields $\frac{\| P - Q \|^2}{(|\lambda| - p_2)^{\frac{1}{2}}} + (|\lambda|^2| - p^2)^{\frac{1}{2}}$.

Using the operator inequality

$2 \| P - Q \| \sqrt{|\lambda|^2 - p^2} \leq \| P - Q \|^2 + |\lambda|^2 - p^2$

in the inequality gives us $\| P - Q \| \leq \frac{w(P - Q)|\lambda|}{\sqrt{|\lambda|^2 - p^2}}$.

On fixing the original $\lambda_1, \lambda_2, p_1, p_2$ into the resultant inequality and applying an arbitrary $X \in B(H)$, we get

$$\| PX - XQ \| \leq \frac{w(PX - XQ)|\lambda_1 - \lambda_2|}{\sqrt{|\lambda_1 - \lambda_2|^2 - |p_1 - p_2|^2}}$$

$\| PX - XQ \| \sqrt{|\lambda_1 - \lambda_2|^2 - |p_1 - p_2|^2} \leq w(PX - XQ)|\lambda_1 - \lambda_2| \| PX - XQ \|^2 |\lambda_1 - \lambda_2| - |p_1 - p_2|^2 \leq w^2(PX - XQ)|\lambda_1 - \lambda_2|^2 \| PX - XQ \|^2 - \frac{|p_1 - p_2|^2}{|\lambda_1 - \lambda_2|^2} \leq w^2(PX - XQ)$

Lemma 3.0.6.

$P, Q \in P_0(H)$ be non-zero orthogonal projections onto H_1, H_{11}, subspaces of H^n and $\lambda_1, \lambda_2 \in C\backslash\{0\}$, such that $|\lambda_1 - \lambda_2| > p > 0$, then

$\| PX - XQ \| \leq w(PX - XQ) \leq \frac{2||\lambda| w(P - Q)}{(|\lambda|^2| - p^2)^{\frac{1}{2}}}$.

Proof.

It is immediate that

$$\begin{aligned} \|((PX - XQ) - (\lambda_1 - \lambda_2))x\|^2 &= \langle (PX - XQ) - (\lambda_1 - \lambda_2), (PX - XQ) - (\lambda_1 - \lambda_2)\rangle \\ &= \| ((PX - XQ)x \|^2 - 2Re[(\lambda_1 - \lambda_2)\langle (PX - XQ)x,x\rangle] + |\lambda_1 - \lambda_2|^2 \leq p^2 \\ &\Rightarrow \| ((PX - XQ)x \|^2 + |\lambda_1 - \lambda_2|^2 \leq 2Re[(\lambda_1 - \lambda_2)\langle (PX - XQ)x,x\rangle] + p^2 \\ &\leq 2|\lambda_1 - \lambda_2||\langle (PX - XQ)x,x\rangle| + p^2 \end{aligned}$$

thus by taking the supremum over all the unit vectors $x \in H^n$, we get

$\| (PX - XQ)x \|^2 + |\lambda_1 - \lambda_2|^2 \leq 2w(PX - XQ) |\lambda_1 - \lambda_2| + p^2$

but the reverse of the inequality holds for the norm of the operator $PX - XQ$ thus;

$\| PX - XQ \| + | \lambda_1 - \lambda_2 |^2 \geq 2 \| PX - XQ \| \, | \lambda_1 - \lambda_2 |$

so $\| PX - XQ \| - w(P - Q) \leq \frac{1}{2} . \frac{p^2}{| \lambda_1 - \lambda_2 |^2}$.

For an arbitrary $X \in B(H)$, then the operator $PX - XQ$ also obeys the inequality

$\| PX - XQ \| - w(PX - XQ) \leq \frac{1}{2} . \frac{p^2}{| \lambda_1 - \lambda_2 |^2}$

Theorem 3.0.7.

Let $P, Q \in P_0(H)$ be as given in the lemma 4.33 above, then

$$\| PX - XQ \|^2 - w(PX - XQ)^2(T) \leq \frac{2p^2}{| \lambda_1 - \lambda_2 | + \sqrt{\lambda_1 - \lambda_2 - p^2}} w(PX - XQ)$$

for an arbitrary $X \in B(H)$

Proof.

From the above lemma 4.43, we have that $\| (PX - XQ)x \|^2 + |\lambda_1 - \lambda_2|^2 \leq 2Re[\lambda_1 - \lambda_2 \langle (PX - XQ)x, x \rangle] + p^2$ for $x \in H^n$ and $\| x \| = 1$. We then divide this inequality by $|\lambda_1 - \lambda_2| |\langle (PX - XQ)x, x \rangle| \geq 0$ so that we get

$$\frac{\|(PX - XQ)x\|^2}{|\lambda_1 - \lambda_2| |\langle (PX - XQ)x, x \rangle|} \leq \frac{2Re[\lambda_1 - \lambda_2 \langle (PX - XQ)x, x \rangle]}{|\lambda_1 - \lambda_2| |\langle Tx, x \rangle|} + \frac{p^2}{|\lambda_1 - \lambda_2| |\langle Tx, x \rangle|} - \frac{|\lambda_1 - \lambda_2|}{|\langle Tx, x \rangle|}$$

, for a unit vector $x \in H^n$. Now we subtract $\frac{|\langle (PX - XQ)x, x \rangle|}{|\lambda_1 - \lambda_2|}$ from both sides of the inequality above, we have

$$\frac{\| (PX - XQ)x \|^2}{|\lambda_1 - \lambda_2| |\langle (P - Q)x, x \rangle|} - \frac{|\langle (PX - XQ)x, x \rangle|}{|\lambda_1 - \lambda_2|}$$
$$\leq \frac{2Re[\lambda_1 - \lambda_2 \langle (PX - XQ)x, x \rangle]}{|\lambda_1 - \lambda_2| |\langle (P - Q)x, x \rangle|} + \frac{p^2}{|\lambda_1 - \lambda_2| |\langle (PX - XQ)x, x \rangle|} - \frac{|\langle (PX - XQ)x, x \rangle|}{|\lambda_1 - \lambda_2|} - \frac{|\lambda_1 - \lambda_2|}{|\langle (PX - XQ)x, x \rangle|}$$
$$= \frac{2Re[\lambda_1 - \lambda_2 \langle (PX - XQ)x, x \rangle]}{|\lambda_1 - \lambda_2| |\langle (P - Q)x, x \rangle|} - \frac{|\lambda_1 - \lambda_2| - p^2}{|\lambda_1 - \lambda_2| |\langle (PX - XQ)x, x \rangle||} - \frac{|\langle (PX - XQ)x, x \rangle|}{|\lambda_1 - \lambda_2|}$$
$$= \frac{2Re[\lambda_1 - \lambda_2 \langle (PX - XQ)x, x \rangle]}{|\lambda_1 - \lambda_2| |\langle (P - Q)x, x \rangle|} - \left(\frac{|\sqrt{|\lambda_1 - \lambda_2|^2 - p^2}}{|\lambda_1 - \lambda_2| |\langle (PX - XQ)x, x \rangle|} - \frac{\sqrt{|\langle (PX - XQ)x, x \rangle|}}{|\lambda_1 - \lambda_2|} \right)^2$$
$$\frac{|\langle (P - Q)x, x \rangle|}{|\lambda_1 - \lambda_2|} - 2 \frac{\sqrt{|\lambda_1 - \lambda_2|^2 - p^2}}{|\lambda_1 - \lambda_2|}$$

Because

$Re[|\lambda_1 - \lambda_2| \langle (P - Q)x, x \rangle] \leq |\lambda_1 - \lambda_2| |\langle (PX - XQ)x, x \rangle|$

and

$\left(\sqrt{|\lambda_1 - \lambda_2|^2 - p^2} \, |\lambda_1 - \lambda_2| |\langle (PX - XQ)x, x \rangle| - \sqrt{\langle (PX} \right.$

is positive, therefore

$$\frac{\| (PX - XQ)x \|^2}{|\lambda_1 - \lambda_2| \langle (PX - XQ)x, x \rangle} \| (PX - XQ)x \|^2$$
$$\leq |\langle (PX - XQ)x, x \rangle|^2 + 2|\langle (PX - XQ)x, x \rangle| (|\lambda_1 - \lambda_2| - \sqrt{|\lambda_1 - \lambda_2|^2 - p^2})$$

Now by taking the supremum over all $x \in H^n$, $\| x \| = 1$, we get

$$\| (PX - XQ) \|^2 \leq \sup \left\{ |\langle (PX - XQ)x, x \rangle|^2 + 2|\langle (PX - XQ)x, x \rangle| \left(|\lambda_1 - \lambda_2| - \sqrt{|\lambda_1 - \lambda_2|^2 - p^2} \right) \right\}$$
$$\leq \sup\{|\langle (PX - XQ)x, x \rangle|^2\} + 2\left(|\lambda_1 - \lambda_2| - \sqrt{|\lambda_1 - \lambda_2|^2 - p^2}\right) \sup\{|\langle (PX - XQ)x, x \rangle|\}$$
$$= w2((PX - XQ)) + 2(|\lambda_1 - \lambda_2| - \sqrt{|\lambda_1 - \lambda_2|^2 - p^2}) w((PX - XQ)$$

Corollary 3.0.8.

Let $\alpha, \beta \in \mathbb{C}$ with $Re(\alpha, \beta) > 0$. Suppose that $P, Q \in P_0 B(H)$ such that $Re\langle \beta x - Px, Px - \alpha x \rangle \geq 0$ and $\| P - \lambda_1 \| \leq p_1$ hold for λ_1, p_1, and $x \in H^n$, $\| x \| = 1$ then:

$\| PX - XQ \| - w2(PX - XQ) \leq [|\beta + \alpha| - 2\sqrt{Re(\alpha, \beta)}] w(PX - XQ)$

Conclusions

Authors have shown that the nth power of numerical radius w^n is bounded above by the sums of $\| (PX - XQ) \|$ of even powers. We have also shown that $\| PX - XQ \|^2$ is bounded above by the sums multiples of $(PX - XQ)$.

Conflicts of interest

Authors declare no conflict of interest.

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