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Research Article

Characterization of Properties of Aluthge Transforms in Banach Algebras

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Abstract

Let $\mathcal{H}$ be a complex separable Hilbert space with and let $\mathcal{B}(\mathcal{H})$ be the Banach algebra of all bounded linear operators on $\mathcal{H}$. In the present paper we characterize Aluthge transforms in Banach algebras. We considered classical and maximal numerical ranges of these transforms and finally we give their relationships.

Keywords: Hilbert space; Numerical ranges; Aluthge transforms; Banach algebra.

Introduction

It is known that the numerical range $W(T)$ of T is the subset $W(T)=\{\langle Tx, x\rangle: x \in H, \|x\| = 1\}$ of the complex plane $\mathbb{C}$ [1]. It is known that $W(T)$ is always convex and the closure $\overline{W(T)}$ of $W(T)$ contains $\sigma(T)$. On the other hand, essential numerical range of T is the subset $W_e(T) = \{\lambda \in \mathbb{C}:$ there exists a unit vector sequence $\{x_n\} \subset \mathcal{H}$ such that x_n converges weakly to 0, $\langle Tx_n, x_n\rangle \to \lambda\}$. It is known [2] that $W_e(T)$ is also always non-empty closed and convex and contains $\sigma_e(T)$. In [3], we also have $W_e(T) = \cap\{\overline{W(T+K)}: K \in \mathcal{K}(\mathcal{H})\}$.

In [4] the author introduced the concept of the maximal numerical range $W_0(T)$ of T to consider the norm of a derivation on $\mathcal{B}(\mathcal{H})$. The maximal numerical range of T is defined to be the subset $W_0(T) = \{\lambda \in \mathbb{C}:$ there exists a unit vector sequence $\{x_n\} \subset \mathcal{H}$ such that $\langle Tx_n, x_n\rangle \to \lambda, \|Tx_n\| \to \|T\|\}$. It was proved in [5] that $W_0(T)$ is a non-empty closed and convex subset of $\mathbb{C}$. We note that $W_0(T)$ does not have translation property by scalar, that is $W_0(T+\lambda) \neq W_0(T)+\lambda$. In particular, we know that for any $\lambda_1 \neq \lambda_2$ in $\mathbb{C}$, $W_0(T+\lambda_1) \bigcap W_0(T+\lambda_2) = \emptyset$. For a more detailed discussion of the maximal numerical range we refer to [6]. For a subset Δ of $\mathbb{C}$, we denote by $\Delta^\wedge$ the closed convex hull of Δ. If $T \in \mathcal{B}(\mathcal{H})$ with a polar decomposition $T=U|T|$, then the Aluthge transform $\widetilde{T}$ and *-Aluthge transform $\widetilde{T}^{(*)}$ are defined by $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ and $\widetilde{T}^{(*)} = |T^*|^{1/2}U|T^*|^{1/2}$ respectively [7]. Note that both $\widetilde{T}$ and $\widetilde{T}^{(*)}$ are independent of the choice of the partial isometry U in the polar decomposition of T. Recently, T, $\widetilde{T}$ and $\widetilde{T}^{(*)}$ have been studied by many authors [8]. In this note, we consider the essential numerical range and the maximal numerical range of T, $\widetilde{T}$ and $\widetilde{T}^{(*)}$. We prove that $W_e(\widetilde{T}) = W_e(\widetilde{T}^{(*)}) \subseteq W_e(T)$ and $W_0(\widetilde{T}+\lambda) = W_0(\widetilde{T}^{(*)}+\lambda)$ for all $\lambda \in \mathbb{C}$.

Let $T \in \mathcal{B}(\mathcal{H})$ and $T=U|T|$ be the polar decomposition of T, then [9] we have $N(T)= N(|T|)=N(U)$. In terms of the orthogonal decomposition $\mathcal{H} = N(T) \bigoplus N(T)^\perp$ of $\mathcal{H}$, T has the following matrix form $T = \begin{pmatrix} 0 & A \\ 0 & B \end{pmatrix}$ for some bounded linear operators A from $N(T)^\perp$ to $N(T)$ and B on $N(T)^\perp$. Now it is known [10] that

$$U = \begin{pmatrix} 0 & U_1 \\ 0 & U_2 \end{pmatrix} \quad \text{and} \quad T^*T = \begin{pmatrix} 0 & 0 \\ 0 & A^*A+B^*B \end{pmatrix}$$

for some operators U_1 and U_2. By a simple calculus, $\widetilde{T}$ has the following matrix form $\widetilde{T} = \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}$, where $X = (A^*A+B^*B)^{1/4}U_2(A^*A+B^*B)^{1/4}$ on $N(T)^\perp$.

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It is known that $\widetilde{T}^{(*)} = U\widetilde{T}U^*$ and $\widetilde{T} = U^*\widetilde{T}^{(*)}U$. Note that U is a unitary operator from $N(T)^{\perp}$ to $N(T^*)^{\perp}$, then there is a unitary operator U_0 from $N(T)^{\perp}$ onto $N(T^*)^{\perp}$ suchthat $U = \begin{pmatrix} 0 & 0 \\ 0 & U_0 \end{pmatrix}$ from $N(T) \bigoplus N(T)^{\perp}$ to $N(T^*) \bigoplus N(T^*)^{\perp}$. It follows that $\widetilde{T}^{(*)} = \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix}$ with respect to the space decomposition $\mathcal{H} = N(T^*) \bigoplus N(T^*)^{\perp}$, where $Y = U_0 X U_0^*$.

Research methodology

Proposition 2.1

Let $T \in \mathcal{B}(\mathcal{H})$. Then for all $K \in \mathcal{K}(\mathcal{H})$, $\widetilde{T+K} - \widetilde{T} \in \mathcal{K}(\mathcal{H})$.

Proof

Since $(T+K)^*(T+K) = T^*T + K_1$ for some $K_1 \in \mathcal{K}(\mathcal{H})$, we have $|T+K|^2 = |T|^2 + K_1$. It follows that $(\pi(|T+K|))^2 = (\pi(|T|))^2$ and $\pi(|T+K|) = \pi(|T|)$, which implies that $|T+K| - |T| \in \mathcal{K}(\mathcal{H})$ and again $|T+K|^{1/2} - |T|^{1/2} \in \mathcal{K}(\mathcal{H})$.

Let $T=U|T|$ and $T+K = V|T+K|$ be the decomposition of T and $T+K$ respectively. Note that $K = V|T+K| - U|T| \in \mathcal{K}(\mathcal{H})$, then $V(|T+K| - |T|) + (V-U)|T| \in \mathcal{K}(\mathcal{H})$, and therefore $(V-U)|T| \in \mathcal{K}(\mathcal{H})$ since $V(|T+K| - |T|) \in \mathcal{K}(\mathcal{H})$. If $|T|$ is invertible, then $(V-U)|T|^{1/2} \in \mathcal{K}(\mathcal{H})$. Otherwise, put $f(t) = t^{1/2}$, $t \in [0, |T|]$. We may choose a sequence of polynomials $P_n(t)$ with $P_n(0)=0$ such that $\lim_{n\to\infty} \|P_n - f\| = 0$ in $C[0,|T|]$ by Stone–Weierstrass Theorem. It is clear $(V-U)P_n(|T|) \in \mathcal{K}(\mathcal{H})$ for all n. It follows that $(V-U)|T|^{1/2} \in \mathcal{K}(\mathcal{H})$ and therefore $|T|^{1/2}(V-U)|T|^{1/2} \in \mathcal{K}(\mathcal{H})$. Then

$$\widetilde{T+K} - \widetilde{T} = |T+K|^{1/2}V|T+K|^{1/2} - |T|^{1/2}U|T|^{1/2}$$
$$= |T+K|^{1/2}V(|T+K|^{1/2} - |T|^{1/2}) + (|T+K|^{1/2}(V-U)|T|^{1/2})$$
$$+ (|T+K|^{1/2} - |T|^{1/2})U|T|^{1/2} \in \mathcal{K}(\mathcal{H})$$

since $|T+K|^{1/2} - |T|^{1/2} \in \mathcal{K}(\mathcal{H})$.

Proposition 2.2

Let $T \in \mathcal{B}(\mathcal{H})$. *Then* $W_e(\widetilde{T}) \subseteq W_e(T)$.

Proof

It is known that $W_e(T) = \bigcap\{\overline{W(T+K)} : K \in \mathcal{K}(\mathcal{H})\}$, that is, $W_e(T) = W_e(T+K)$ for all $K \in \mathcal{K}(\mathcal{H})$. By [7], we have $W_e(\widetilde{T}) = W_e(\widetilde{T+K}) \subseteq \overline{W(\widetilde{T+K})} \subseteq \overline{W(T+K)}$.

Thus $W_e(\widetilde{T}) \subseteq \bigcap\{\overline{W(T+K)} : K \in \mathcal{K}(\mathcal{H})\} = W_e(T)$.

Lemma 2.3

Let $T \in \mathcal{B}(\mathcal{H})$. *Then* $\sigma_e(\widetilde{T}) = \sigma_e(\widetilde{T}^{(*)})$.

Proof

Recall that $\sigma_e(T) = \sigma_{le}(T) \bigcup \sigma_{re}(T)$. We first prove that $\sigma_{le}(\widetilde{T}) \setminus \{0\} = \sigma_{le}(\widetilde{T}^{(*)}) \setminus \{0\}$.

Suppose $\lambda \in \sigma_{le}(\widetilde{T}) \setminus \{0\}$. Then there exists a unit vector sequence $\{x_n\}$ in $\mathcal{H}$ such that $\{x_n\}$ converges weakly to zero and $\lim_{n\to\infty} \|(\widetilde{T} - \lambda)x_n\| = 0$, that is, $\lim_{n\to\infty} \|(U^*\widetilde{T}^{(*)}U - \lambda)x_n\| = 0$. In fact, we may choose $\{x_n\}$ in $N(T)^{\perp}$ such that $\|Ux_n\| = 1$ for all integer n. It follows that $\lim_{n\to\infty} \|(\widetilde{T}^{(*)} - \lambda)Ux_n\| = \lim_{n\to\infty} \|(UU^*\widetilde{T}^{(*)} - \lambda)Ux_n\| \leq \lim_{n\to\infty} \|(\widetilde{T} - \lambda)x_n\| = 0.$

Note that $U x_n$ converges weakly to zero, then $\lambda \in \sigma_{le}(\widetilde{T}^{(*)})$.On the other hand, we can obtain $\sigma_{le}((\widetilde{T})^{(*)}) \setminus \{0\} \subseteq \sigma_{le}(\widetilde{T}) \setminus \{0\}$ by a similar method. Then $\sigma_{le}(\widetilde{T}) \setminus \{0\} = \sigma_{le}(\widetilde{T}^{(*)}) \setminus \{0\}$.It also follows that $\sigma_{re}(\widetilde{T}) \setminus \{0\} = \sigma_{re}(\widetilde{T}^{(*)}) \setminus \{0\}$ by the fact that $\sigma_{re}(A) = \sigma_{le}(A^*)$ for any $A \in \mathcal{B}(\mathcal{H})$. Then $\sigma_e(\widetilde{T}) \setminus \{0\} = \sigma_e(\widetilde{T}^{(*)}) \setminus \{0\}$. Next we show that $0 \in \sigma_e(\widetilde{T})$ if and only if $0 \in \sigma_e(\widetilde{T}^{(*)})$. This is equivalent to show that $\widetilde{T}$ is Fredholm if and only if $\widetilde{T}^{(*)}$ is. Note that $\widetilde{T}$ (resp. $\widetilde{T}^{(*)}$) is Fredholm if and only if U and $|T|^{1/2}$(resp. $|T^*|^{1/2}$) are Fredholm. It follows that $\widetilde{T}$ is Fredholm if and only if $\widetilde{T}^{(*)}$ is by the facts that $\widetilde{T} = U^*\widetilde{T}^{(*)}U$ and $\widetilde{T}^{(*)} = U\widetilde{T}U^*$. Above all, we have $\sigma_e(\widetilde{T}) = \sigma_e(\widetilde{T}^{(*)})$.

Theorem 2.4

Let $T \in \mathcal{B}(\mathcal{H})$. *Then* $W_e(\widetilde{T}) = W_e(\widetilde{T}^{(*)})$.

Proof

We recall that $\widetilde{T} = \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}$ and $\widetilde{T}^{(*)} = \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix}$, where X and Y are unitarily equivalent. Then we easily have $W_e(X) = W_e(Y)$

$W_e(X) \subseteq W_e(\widetilde{T}) \subseteq (W_e(X) \cup \{0\})^{\wedge}$ and $W_e(Y) \subseteq W_e(\widetilde{T}^{(*)})$

To complete the proof, it is sufficient to prove that $0 \in W_e(\widetilde{T})$ if and only if $0 \in W_e(\widetilde{T}^{(*)})$. Suppose $0 \in W_e(\widetilde{T})$. If $0 \notin W_e(\widetilde{T}^{(*)})$, then $W_e(\widetilde{T}^{(*)}) = W_e(Y)$ and $0 \notin \sigma_e(\widetilde{T}^{(*)})$. By Proposition 2.1, we also have $0 \notin \sigma_e(\widetilde{T})$. It follows that $N(T)$ is finite-dimensional and X is Fredholm. Then $W_e(\widetilde{T}) = W_e(X) = W_e(Y) = W_e(\widetilde{T}^{(*)})$. This is a contradiction. Thus, $0 \in W_e(\widetilde{T}^{(*)})$. Conversely, if $0 \in W_e(\widetilde{T}^{(*)})$, we similarly have $0 \in W_e(\widetilde{T})$.

Results and discussion

In this section, we give the main results of our study. We begin with the following Lemma.

Lemma 3.1

Let $T \in \mathcal{B}(\mathcal{H})$. Then $W_0(T^*) = (W_0(T))^* = \{\lambda : \overline{\lambda} \in W_0(T)\}$.

Proof

We may assume that $\|T\| = \|T^*\| = 1$. Suppose $\lambda \in W_0(T)$. Then there exists a unit vector sequence $\{x_n\}$ in $\mathcal{H}$ such that $\lim_{n\to\infty} \|Tx_n\| = 1$ and $\lim_{n\to\infty}\langle Tx_n, x_n\rangle = \lambda$. Now we have

$$\lim_{n\to\infty} |\langle x_n, x_n\rangle - \langle Tx_n, Tx_n\rangle| = \lim_{n\to\infty} |\langle (1 - T^*T)x_n, x_n\rangle| = 0,$$

which implies that $\lim_{n\to\infty} \|(1 - T^*T)^{1/2}x_n\|^2 = 0$. Then $\lim_{n\to\infty} \|(1 - T^*T)x_n\| = 0$ and $\lim_{n\to\infty} \|T^*Tx_n\| = 1$. In particular, $\lim_{n\to\infty} \|Tx_n\| = 1$. Hence,

$$\lim_{n\to\infty} |\langle T^*Tx_n, Tx_n\rangle - \langle x_n, Tx_n\rangle| = \lim_{n\to\infty} |\langle (T^*T - 1)x_n, Tx_n\rangle|$$
$$\le \lim_{n\to\infty} \|(T^*T - 1)x_n\|\|Tx_n\| = 0.$$

We now have $\lim_{n\to\infty}\langle T^*Tx_n, Tx_n\rangle = \lim_{n\to\infty}\langle x_n, Tx_n\rangle = \overline{\lambda}$. Here put $y_n = Tx_n/\|Tx_n\|$. Then $\{y_n\}$ is a unit vector sequence, $\lim_{n\to\infty} \|T^*y_n\| = 1$ and $\lim_{n\to\infty}\langle T^*y_n, y_n\rangle = \overline{\lambda}$, which implies that $\overline{\lambda} \in W_0(T^*)$. Then $(W_0(T))^* \subseteq W_0(T^*)$. By symmetry, we have $W_0(T^*) = (W_0(T))^*$.

Lemma 3.2

Let $T \in \mathcal{B}(\mathcal{H})$. Then $\|\widetilde{T} - \lambda\| = \|\widetilde{T}^{(*)} - \lambda\|$ for all $\lambda \in \mathbb{C}$.

Proof

Let $\widetilde{T} = \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}$ and $\widetilde{T}^{(*)} = \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix}$ with respect to the space decomposition $\mathcal{H} = N(T) \bigoplus N(T)^{\perp}$ and $H = N(T^*) \bigoplus N(T^*)^{\perp}$ respectively, where $Y = U_0XU_0^*$, and U_0 is unitary.

Let $\lambda \in \mathbb{C}$, then we have $\widetilde{T} - \lambda = \begin{pmatrix} -\lambda & 0 \\ 0 & X-\lambda \end{pmatrix}$, $\widetilde{T}^{(*)} - \lambda = \begin{pmatrix} -\lambda & 0 \\ 0 & Y-\lambda \end{pmatrix}$, and $Y - \lambda = U_0(X - \lambda)U_0^*$. Then $\|X - \lambda\| = \|Y - \lambda\|$.

Case 1 *N(T)*$\neq\{0\}$ and $N(T^*) \neq \{0\}$. Then $\|\widetilde{T} - \lambda\| = \max\{|\lambda|, \|X - \lambda\|\} = \max\{\|\lambda\|, \|Y - \lambda\|\} = \|\widetilde{T}^{(*)} - \lambda\|$.

Case 2 *N(T)*$=\{0\}$ and $N(T^*) = \{0\}$. In this case, $\widetilde{T} - \lambda = X - \lambda,\ \widetilde{T}^{(*)} - \lambda = Y - \lambda$. Clearly $\|\widetilde{T} - \lambda\| = \|X - \lambda\| = \|Y - \lambda\| = \|\widetilde{T}^{(*)} - \lambda\|$.

Case 3 *N(T)*$=\{0\}$ and $N(T^*) \neq \{0\}$. Then $\widetilde{T} - \lambda = X - \lambda$. It follows that $\|\widetilde{T} - \lambda\| = \|X - \lambda\|$. Next we show that $\|\widetilde{T}^{(*)} - \lambda\| = \|Y - \lambda\|$. Otherwise, if $\|Y - \lambda\| < |\lambda|$, then $-\lambda \in \rho(Y - \lambda)$, which implies that Y is invertible. It follows that X is also invertible since X and Y are unitarily equivalent. Thus $\widetilde{T} = X$ is invertible and so are T and T*. However, $N(T^*) \neq 0$. This is a contradiction.
Hence $\|Y - \lambda\| \geq |\lambda|$ and $\|\widetilde{T}^{(*)} - \lambda\| = \|Y - \lambda\|$. Therefore $\|\widetilde{T} - \lambda\| = \|\widetilde{T}^{(*)} - \lambda\|$.

Case 4 $N(T) \neq \{0\}$ and $N(T^*) = \{0\}$. The proof is similar to Case 3.

Remark 3.3

Let U be a non-unitary isometry on $\mathcal{H}$. It is known that $U = U_0 \oplus U_1$ from the von Neumann-Wold Decomposition Theorem, where U_0 is unitary and U_1 is a unilateral shift.

Theorem 3.4

Let $T \in \mathcal{B}(\mathcal{H})$. Then $W_0(\widetilde{T} - \lambda) = W_0(\widetilde{T}^{(*)} - \lambda)$ for all $\lambda \in \mathbb{C}$.

Proof

Let $\lambda \in \mathbb{C}$, we have $\widetilde{T} - \lambda = \begin{pmatrix} -\lambda & 0 \\ 0 & X-\lambda \end{pmatrix}$ and $\widetilde{T}^{(*)} - \lambda = \begin{pmatrix} -\lambda & 0 \\ 0 & Y-\lambda \end{pmatrix}$ with respect to the space decomposition $\mathcal{H} = N(T) \bigoplus N(T)^{\perp}$ and

$\mathcal{H} = N(T^*) \bigoplus N(T^*)^\perp$ respectively, where $X-\lambda$ and $Y-\lambda$ are unitarily equivalent.

Case 1 $N(T)=\{0\}$ and $N(T^*) = \{0\}$. In this case $\widetilde{T} - \lambda = X - \lambda,\ \widetilde{T}^{(*)} - \lambda = Y - \lambda$.

The result follows.

Case 2 $N(T)\neq\{0\}$ and $N(T^*) \neq \{0\}$.

If $\|Y-\lambda\| = \|X-\lambda\| = |\lambda|$, then $W_0(\widetilde{T}-\lambda) = (\{\lambda\} \bigcup W_0(X-\lambda))^\wedge = (\{\lambda\} \bigcup W_0(Y-\lambda))^\wedge = W_0(\widetilde{T}^{(*)}-\lambda)$ by Lemma 3.1.

If $\|X-\lambda\| = \|Y-\lambda\| > |\lambda|$, then $W_0(\widetilde{T}-\lambda) = W_0(X-\lambda) = W_0(Y-\lambda) = W_0(\widetilde{T}^{(*)}-\lambda)$ by Lemma 3.2.

If $\|X-\lambda\| = \|Y-\lambda\| < |\lambda|$, then $W_0(\widetilde{T}-\lambda) = \{-\lambda\} = W_0(\widetilde{T}^{(*)}-\lambda)$ by Lemma 3.1 again.

Case 3 $N(T)=\{0\}$ and $N(T^*) \neq \{0\}$. Then $\widetilde{T} - \lambda = X - \lambda$ and $W_0(\widetilde{T}-\lambda) = W_0(X-\lambda)$. We next prove that $\|Y-\lambda\| > |\lambda|$. Note that $\widetilde{\lambda T} = \lambda\widetilde{T}$. Without loss of generality, we may assume that $\lambda = 1$.

We have $\|\widetilde{T}^{(*)} - 1\| = \|\widetilde{T} - 1\| = \|X-1\| = \|Y-1\| \geq 1$ by Lemma 3.1 and our assumption.

In fact, if $\|\widetilde{T}^{(*)} - 1\| = \|\widetilde{T} - 1\| < 1$, then we have that $\widetilde{T}$ is invertible. Then so is T. This contradicts with the assumption of this case. If $\|Y-1\| = 1$, then $\|\widetilde{T} - 1\| = 1$. Note that U is non-unitary isometry and $|T|^{1/2}$ is injective with dense range since $N(T) = 0$ and $N(T^*) \neq 0$. Then for any unit vector $x \in \mathcal{H}$, we have $|\langle \widetilde{T}x, x\rangle - 1| \leq 1$, which implies that
$$\left| \||T|^{1/2}x\|^2 \left\langle U\frac{|T|^{1/2}x}{\||T|^{1/2}x\|}, \frac{|T|^{1/2}x}{\||T|^{1/2}x\|} \right\rangle - 1 \right| \leq 1.$$
It is clear that
$$\left\langle U\frac{|T|^{1/2}x}{\||T|^{1/2}x\|}, \frac{|T|^{1/2}x}{\||T|^{1/2}x\|} \right\rangle \in W(U).$$

Note that $|T|^{1/2}$ has dense range. Then we can choose a unit vector $x_0 \in H$ such that
$$\left\langle U\frac{|T|^{1/2}x_0}{\||T|^{1/2}x_0\|}, \frac{|T|^{1/2}x_0}{\||T|^{1/2}x_0\|} \right\rangle \in (-1, 0)$$
by Lemma 3.2. It follows that
$$\left| \||T|^{1/2}x_0\|^2 \left\langle U\frac{|T|^{1/2}x_0}{\||T|^{1/2}x_0\|}, \frac{|T|^{1/2}x_0}{\||T|^{1/2}x_0\|} \right\rangle - 1 \right| > 1.$$

This is a contradiction. Hence $|Y-1|>1$ and $W_0(\widetilde{T}^{(*)} - 1) = W_0(Y-1)$ by Lemma 3.2.

We then generally have $W_0(\widetilde{T}^{(*)} - \lambda) = W_0(Y-\lambda) = W_0(X-\lambda) = W_0(\widetilde{T}-\lambda)$ by Lemma 2.

Case 4 $N(T) \neq \{0\}$ and $N(T^*) = \{0\}$. The proof is similar to Case 3.

We recall that an inner derivation determined by $A \in \mathcal{B}(\mathcal{H})$ is defined by $\delta_A(X) = AX - XA$ for all $X \in \mathcal{B}(\mathcal{H})$. Stampfli in 8 gave the norm $\|\delta_A\|$ of δ_A by using of maximal numerical range, that is, $\|\delta_A\| = \inf\{\|T-\lambda\| : \lambda \in \mathbb{C}\}$. By Theorem 3, we have

Theorem 3.5

Let $T \in \mathcal{B}(\mathcal{H})$.

$W_0(T) \subset \overline{W(\widetilde{T})}$.

If $\|T\| = \|\widetilde{T}\|$, *then* $W_0(\widetilde{T}) \subset W_0(T)$.

Proof

Without loss of generality, we may assume that $\|T\| = 1$.

Let $\lambda \in W_0(T)$, then there exists a unit vector sequence of $\{x_n\}$ in $\mathcal{H}$ such that $\lim_{n\to\infty} \|Tx_n\| = 1$ and $\lim_{n\to\infty} \langle Tx_n, x_n\rangle = \lambda$, which implies that $\lim_{n\to\infty} \||T|^{1/2}x_n\| = 1$ and $\lim_{n\to\infty} \|(1-|T|)x_n\| = 0$.
$$\begin{aligned}
&\lim_{n\to\infty} |\langle Tx_n, x_n\rangle - \langle \widetilde{T}|T|^{1/2}x_n, |T|^{1/2}x_n\rangle| \\
&= \lim_{n\to\infty} |\langle Tx_n, x_n\rangle - \langle U|T|x_n, |T|x_n\rangle| \\
&= \lim_{n\to\infty} |\langle Tx_n, (1-|T|)x_n\rangle| \\
&\leq \lim_{n\to\infty} \|Tx_n\| \|(1-|T|)x_n\| = 0.
\end{aligned}$$
Hence It follows that $\lim_{n\to\infty} \langle \widetilde{T}|T|^{1/2}x_n, |T|^{1/2}x_n\rangle = \lambda$. Here put $y_n = |T|^{1/2}x_n/(\||T|^{1/2}x_n\|)$. Then $\{y_n\}$ is a unit vector sequence and $\lim_{n\to\infty} \langle \widetilde{T}y_n, y_n\rangle = \lambda$, and therefore $\lambda \in \overline{W(\widetilde{T})}$.

We have $\|T\| = \|\widetilde{T}\| = 1$. Suppose $\lambda \in W_0(\widetilde{T})$. Then there exists a unit vector sequence of $\{x_n\}$ in $\mathcal{H}$ such that $\lim_{n\to\infty} \|\widetilde{T}x_n\| = 1$ and $\lim_{n\to\infty} \langle \widetilde{T}x_n, x_n\rangle = \lambda$. It easily follows that $\lim_{n\to\infty} \||T|^{1/2}x_n\| = \||T|^{1/2}\| = 1$ and then $\lim_{n\to\infty} \|(1-|T|)x_n\| = 0$. We easily have $\lim_{n\to\infty} \|(1-|T|^3)x_n\| = 0$ also.

Thus $\lim_{n\to\infty} \| T|T|^{1/2} x_n \| = \lim_{n\to\infty} \langle |T|^3 x_n,\ x_n \rangle = 1 = \|T\|$.

On the other hand,

$$\begin{aligned}
&\lim_{n\to\infty} |\langle \widetilde{T} x_n,\ x_n\rangle - \langle T|T|^{1/2} x_n,\ |T|^{1/2} x_n\rangle| \\
&= \lim_{n\to\infty} |\langle U|T|^{1/2} x_n,\ |T|^{1/2} x_n\ \rangle - \langle\ T|T|^{1/2} x_n,\ |T|^{1/2} x_n\ \rangle| \\
&= \lim_{n\to\infty} |\langle (U|T|^{1/2} - U|T||T|^{1/2})\ x_n,\ |T|^{1/2} x_n\rangle| \\
&= \lim_{n\to\infty} |\langle (U|T|^{1/2})\ (1 - |T|)\ x_n,\ |T|^{1/2} x_n\rangle| \\
&\leq \lim_{n\to\infty} \|U|T|^{1/2}\|\ \|(1 - |T|) x_n\|\ \||T|^{1/2} x_n\| = 0.
\end{aligned}$$

Here put $y_n = |T|^{1/2} x_n / (\||T|^{1/2} x_n\|)$.

Then $\{y_n\}$ is a unit vector sequence and $\lim_{n\to\infty} \|T y_n\| = \|T\| = 1$ and $\lim_{n\to\infty} \langle T y_n,\ y_n \rangle = \lambda$. Thus $\lambda \in W_0(T)$.

Conclusions

If we let $\mathcal{H}$ to be a complex separable Hilbert space and we let $\mathcal{B}(\mathcal{H})$ be the Banach algebra of all bounded linear operators on $\mathcal{H}$, we have characterized Aluthge transforms in Banach algebras. We have considered the classical and maximal numerical ranges of these transforms and finally we have given their relationships.

Conflicts of interest

Authors declare no conflict of interest.

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