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Research Article

Prey-Predator Mathematical System Analysis through Lyapunov Function

Judith J. E. J. Ogal, N. B. Okelo

School of Mathematics, Statistics and Actuarial Science,
Jaramogi Oginga Odinga University of Science and Technology,
P. O. Box 210-40601, Bondo-Kenya.

*Corresponding author's e-mail: bnyaare@yahoo.com

Abstract

Pests are very important in crop production especially aphids are important pests which cannot be ignored in agriculture. The damage they cause to these crops as well as loss of yields can be extensive if not contained. However, to contain these pests, it is important to understand its dynamics in relation to its interaction with its natural enemies like the ladybird. In mathematics, the best tool that can be used to understand this prey-predator dynamics is the models which have different variables and parameters that represent the various aspects of the dynamics of the prey-predator system that we are interested in. In this study, we have therefore gone an extra mile to construct sets of mathematical models, by adjusting the function representing the prey-predator interaction.

Keywords: Prey-predator; Dynamics model; Ladybird; Lyapunov Function.

Introduction

In [1] the authors discussed forecasting cereal aphid outbreaks. They modelled the effects of coccinellids, parasitoids and disease. In their model, they used steps of model initialization, data input, hourly temperatures, immigration, development and survival, reproduction and morph determination, predators, output, crop development model and input variable. The advection term represented the predator density movement according to a basic prey taxis assumption: acceleration of predators is proportional to the prey gradient. The prey population reproduced logistically, and the local population interactions followed the Holling Type II function. Their spatially explicit model subdivided the predation process into random movement represented by diffusion, directed movement was described by prey taxis, local prey encounters, and consumption modelled by trophic function. The model enabled studying the effects of large-scale predator spatial activity on population dynamics.

In [2] the authors came up with a logistic model with variable carrying capacity and growth rate affected by cumulative density to study the population dynamics of aphids. In [3], a flux-based model to describe an aphid-parasitoidal system in a closed structured environment has been presented. They applied this approach to the *Aphis gossypii* and to one of its parasitoids, *Lysiphlebu testaceipes* in a melon green house. They developed a model showing host-parasitoidal interactions. The model represented the level of plant infestation as a continuous variable corresponding to the number of plants bearing a given density of pests at a given time. They used partial differential equations to describe the variation of this variable, which was coupled to an ordinary differential equation and a delay-differential equation that described the parasitized host population and the parasitoid population, respectively.

A study in [4] presented a management alternative for the control of pest species through intraguild predation for the spatially homogeneous system. They extended the model to include movement of predator and prey in the spatial context. They considered a spatially homogeneous system and found the conditions for predator and prey to exclude each other, to coexist and for alternative stable states. Other studies in [5-10] presented a general framework

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for age-structured predator-prey systems where individuals were divided into two classes, juveniles and adults, and several possible interactions considered. They used the Rosenzweig-MacArthur prey-predator model which they extended to include delay. They then reduced the initial system of partial differential equations to a system of (neutral) delay differential equations with one or two delays. In this project, we have extended the mathematical background given by Rosenzweig-MacArthur prey-predator model using the work done by [11]. We first formulate two sets of Rosenzweig-MacArthur prey-predator model with one predator and the prey, and then solve them analytically and numerically. The second set of the model seeks to modify and thus give a more accurate analysis of data compared to the first set of the model.

Terminologies

Aphid density: number of aphid per plant.

Carrying capacity: the population size of the species that the environment can sustain indefinitely, given the food, habitat, water and other necessities are available in the environment.

Cumulative density: total number of aphids.

Economic injury level: the smallest number of insects (amount of injury) that will yield losses equal to the insect management cost .

Economic threshold: the pest density at which management action should be taken to prevent an increasing pest population from reaching the economic injury level.

Fecundity: the actual reproductive rate of an organism or population, measured by the number of gametes (eggs), seed set or asexual propagules

Integrated pest management: is an effective and environmentally sensitive approach to pest management that relies on a combination of common-sense practices.

Mortality: death rate.

Oviposition: laying eggs.

Voracity: eagerness to consume great quantities of food.

Research methodology

The varying nature of the soluble nitrogen in the host plants, which is a good indicator of host quality, explains the assumption of the varying carrying capacity. On the other hand, in the cumulative density model incorporated above, the assumption is that the cumulative density is the regulatory term that slows down the instantaneous rate of increase. This is pegged on the assumption that it is the sum of the number of individuals multiplied by their life span, which determines the slowing down of the instantaneous rate of increase. Thus it could influence food quality and hence slow down population rate of increase. The advantage of the Kindlmann model [6] is that it gives the most flexible model. However, its limitation is that there is the problem of how to measure the time varying carrying capacity. The focus of this project is predation as a means of pest control. We particularly look at Coccinellidae (ladybird beetles) as our predator. Many studies have been conducted on ladybirds as aphid predators because they are visible and also have economic importance to a variety of crops. The larvae and adults of coccinellids feed on the same type of prey species and occur in identical habitats [3]. The lifetime fecundity of Coccinellid varies greatly between species and may range from slightly more than 100 to more than 1500 eggs per female. Developmental times varies greatly between species and is influenced by temperature, the amount of food consumed and prey species [5].

Results and discussions

Consider the system

$$\frac{dN}{dt} = N(r-b)\left(1-\frac{N}{K}\right) - \frac{aNP}{D+ahN} \qquad 1(a)$$

$$\frac{dP}{dt} = \mu P - \mathrm{e}\frac{aNP}{D+ahN}$$

and

$$\frac{dN}{dt} = N(r-b)\left(1-\frac{N}{K}\right) - \frac{aNP}{D+ahN+P} \qquad 1(b)$$

$$\frac{dP}{dt} = \mu P - \mathrm{e}\frac{aNP}{D+ahN+P}$$

Without loss of generality we simplify the models by taking *ah* = 1. Hence the above equations 1(a) and (b) respectively in respectively can be written as indicated below.

$$\begin{aligned}\frac{dN}{dt} &= N(r-b)\left(1-\frac{N}{K}\right) - \frac{aNP}{D+N}\\ \frac{dP}{dt} &= \mu P - \mathrm{e}\frac{aNP}{D+N}\end{aligned} \qquad (2)$$

$$\begin{aligned}\frac{dN}{dt} &= N(r-b)\left(1-\frac{N}{K}\right) - \frac{aNP}{D+N+P}\\ \frac{dP}{dt} &= \mu P - \mathrm{e}\frac{aNP}{D+N+P}\end{aligned} \qquad (3)$$

Where, $N>0$ and $P>0$, respectively. This implies that allthe parameters in the model are positive. We then perform non-dimensionalization to reduce the number of parameters in the model in equation (2) and (3) by reducing $\bar{t}$, $\bar{N}$ and $\bar{P}$ into non-dimensional form using,

$$t = \frac{\bar{t}}{r}, N = \bar{N}K, P = \bar{P}eK.$$

Then, further by setting the parameters$\bar{a} = \frac{aek}{r}$, $\bar{\mu} = \frac{\mu}{r}$,$\bar{K} = \frac{K}{e}$ then dropping the sign, we find that the equations (2) and (3) take the form in equations (4) and (5) respective

$$\frac{dN}{dt} = N\left(1-\frac{b}{r}\right)(1-N) - \frac{aNP}{D+N} \qquad 4(a)$$

$$\frac{dP}{dt} = \mu P - \frac{aNP}{D+N} \qquad 4(b)$$

$$\frac{dN}{dt} = N\left(1-\frac{b}{r}\right)(1-N) - \frac{aNP}{D+N+P} \qquad 5(a)$$

$$\frac{dP}{dt} = \mu P - \frac{aN\ P}{D+N+P} \qquad 5(b)$$

$N(0)>0$ and $P(0)>0$, respectively.

The parameters in our analysis of the prey-predator interaction, which we have gotten from a previous study, give us fixed values for these parameters. However, the time span and the densities of the prey/ predator vary. The variation of initial prey's density, *N*, or the initial predator's density, *P*, have an effect on the both the prey and predators population. We explore four possible variations illustrated as (i), (ii), (iii) and (iv), below:

N-small, P-large

When the number of predators largely exceeds that of prey, the result is a decrease in prey population and a decrease in predator population. This is because more predators will be depending on a small number of prey for food, implying that the predator's consumption rate is higher than the prey's growth (increase) rate.

N-large, P-large

Given that, the number of both preys and predators are both large. This results into an increase in the population of both the prey and the predator. The number of predator is directly proportional to the number of prey, so when the prey increases the predators also increases because of the dependence.

N-small, P-small

When the number of the number of both preys and predators is small, there will be a decrease in the population of both the prey and the predator. The number of predator is directly proportional to the number of prey, so when the prey decreases the predators also decreases because of the dependence.

N-large, P-small

When the prey largely outnumbers the predators, the result is an increase in prey population and an increase in predator population. This is because the number of prey on which the predators feed is already high, the growth rate of predators increase because there is enough prey to feed on. These variations on the prey populations and predator numbers in the prey-predator model have an oscillatory character.

Equilibrium points of the model

We determine the conditions for the existence of equilibrium points of the two systems of equations. The two sets of equations in Case I and that in Case II are almost similar, therefore the analysis done below will apply to both. The equilibrium point of $E_0(0, 0)$ is trivial, therefore we will not dwell on it. When we equate equations (4) and (6) to zero, we find that the system has three equilibria, that is;$E_1(1, 0)$,$E_2(0, 1)$ and$E_3(1, 1)$.

Existence of $E_1(\bar{N}, 0)$ *with* $\bar{N}>0$.

Let$P= 0$. Equation (4) gives:

$$N\left(1-\frac{b}{r}\right)(1-N) = 0$$

From this we have $N=1$, which implies that $N = K$ .Thus in the absence of predator *P*, the prey population N increases until it reaches the carrying capacity K.

Existence of E_2 $(0, \bar{P})$ *with* $P > 0$.

Let N=0. Equation (6) gives:

$\mu P = 0,$

Since $P > 0,$ it implies $\mu = 0$. This means that the growth rate of the predator is zero. If the growth rate is zero, there can be no increase in predator's population. Therefore the predator's population will tend to extinction when there is no prey.

Existence of E_3 $(\bar{N}, \bar{P})$ *with* $\bar{N}>0$ *and* $\bar{P}>0$.

Equation (4) gives:

$$N\left(1 - \frac{b}{r}\right)(1 - N) - \frac{aNP}{D + N} = 0$$

and

$$\mu P - \frac{aNP}{D + N} = 0$$

From these we have,

$$\bar{N} = \frac{\mu}{a}(D + N)$$

And

$$\bar{P} = \frac{1}{a}\left[(D + N)\left(1 - \frac{b}{r}\right)\left(1 - \frac{\mu}{a}(D + N)\right)\right]$$

Thus, $E_3(\bar{N},\bar{P})$=

$$E_3\left(\frac{\mu}{a}(D + N, \frac{1}{a}\left[(D + N)\left(1 - \frac{b}{r}\right)\left(1 - \frac{\mu}{a}(D + N)\right)\right]\right)$$

This exists if, $b < r$ and $\mu < \frac{a}{D+N}$. Therefore the
i.
equilibrium exists if,

$b < r$ (4)

$\mu < \frac{a}{D+N}$ (5)

Condition (4) implies that prey's cumulative density b, which is the regulatory term, must be less than its growth rate r, whereas condition (5) implies that, $\mu < \frac{a}{D+N}$, that is, the predator's mortality rate must be less than the quotient of the maximum killing rate the sum of saturation-constant and the population of the prey. Therefore, E_3 $(\bar{N},\bar{P})$ exists only if conditions (4) and (5) are in place.

Local stability of the equilibrium points

The system of equations for our model is nonlinear therefore we use the Jacobian matrix which enables us to linearize the system and determine the characteristic equation. We get the characteristic equation by calculating $|J - \lambda I,|$ where J is the Jacobian matrix and λI the identity matrix. We then find the roots of the characteristic function which enables us to determine the stability of the equilibrium solution.

The stability of each equilibrium point is studied by computing the Jacobian matrix and finding the eigenvalues evaluated at each equilibrium point. We only focus on finding the local stabilities for the non-trivial equilibrium points. If all real eigenvalues are negative then the equilibrium point is stable. If there is a positive eigenvalue or an eigenvalue with a positive real part, then the equilibrium is unstable. From equations (4), the Jacobian matrix is given by

$$J(E_i) = \begin{bmatrix} \frac{\partial f_1}{\partial N} & \frac{\partial f_1}{\partial P} \\ \frac{\partial f_2}{\partial N} & \frac{\partial f_2}{\partial P} \end{bmatrix}$$

When we work out the values of each individual element in the Jacobian matrix we get,

$$J(E_i) = \begin{bmatrix} A^{**} & -\frac{aN}{(D+N)} \\ -\frac{aP}{(D+N)^2} & \mu - \frac{aN}{(D+N)} \end{bmatrix} \quad (8)$$

Where the value of,

$$A^{**} = \left(1 - \frac{b}{r}\right)(1 - 2N) - \frac{aP}{(D + N)^2}$$

The local stability for each equilibrium point is analyzed as follow:

E_1 $(\bar{N}, 0)$ = $E_1(1, 0)$. The Jacobian matrix evaluated at E_1 gives

$$J\,(E_1) = \begin{bmatrix} \left(\frac{b}{r} - 1\right) & -\frac{aN}{(D+N)} \\ 0 & \mu - \frac{a}{(D+N)} \end{bmatrix} \quad (9)$$

We then take the determinant of the matrix and subtract the identity matrix λI to obtain

$$|J(1, 0) - \lambda I| = \begin{vmatrix} \left(\frac{b}{r} - 1\right) - \lambda & -\frac{aN}{(D + N)} \\ 0 & \mu - \frac{a}{(D + N)} - \lambda \end{vmatrix}$$

The eigenvalues of the matrix $J(E_1)$ are $\lambda_1 = \left(\frac{b}{r} - 1\right)$ and $\lambda_2 = \mu - \frac{a}{(D + N)}$.

These eigenvalues are negative if, $b < r$ and $\mu(D + N) < a$. If eigenvalues λ_1 and λ_2 are both negative, the equilibrium is stable. If $b < r$ and $a < \mu(D + N)$ then at least one of the two eigenvalues is positive. If at least one of the eigenvalues is positive, the equilibrium becomes unstable.

ii. $E_2(0,\bar{P})=E_2(0,0)$

The Jacobian matrix evaluated at E_2 is

$$J(E_2) = \begin{bmatrix} \left(1-\frac{b}{r}\right) & 0 \\ 0 & \mu \end{bmatrix} \qquad (10)$$

Taking the determinant of the matrix and subtracting the identity matrix λI we obtain

$$|J(0,0)-\lambda I| = \begin{vmatrix} \left(1-\frac{b}{r}\right)-\lambda & 0 \\ 0 & \mu-\lambda \end{vmatrix}$$

The eigenvalues of the matrix $J(E_2)$ are $\lambda_1 = \left(1-\frac{b}{r}\right)$ and $\lambda_2 = \mu$. Since, $\lambda_2 = \mu$, and from existence of equilibrium points, we found out that $\mu = 0$,implying that $\lambda_2 = 0$.We also know that $b < r$ which means that the eigenvalue λ_1 is positive, hence the equilibrium is unstable.

$E_3(\bar{N},\bar{P})=$

$$E_3\left[\frac{\mu}{a}(D+N),\frac{1}{a}\left\{(D+N)\left(1-\frac{b}{r}\right)\left(1-\frac{\mu}{a}(D+N)\right)\right\}\right]$$

The Jacobian matrix evaluated at E_3 is

$$J(E_3) = \begin{bmatrix} A_1^* & \mu \\ -B_1^* & 0 \end{bmatrix} \qquad (11)$$

Where

$$A_1^* = \left(1-\frac{b}{r}\right)\left(1-\frac{2\mu}{a}(D+N)\right)-\frac{1}{D+N}\left[(D+N)\left(1-\frac{b}{r}\right)\left(1-\frac{\mu}{a}(D+N)\right)\right]$$

$$B_1^* = -\frac{1}{D+N}\left[(D+N)\left(1-\frac{b}{r}\right)\left(1-\frac{\mu}{a}(D+N)\right)\right]$$

The determinant of the matrix is found by

$$\left|J\left\{\frac{\mu}{a}(D+N),\frac{1}{a}\left[(D+N)\left(1-\frac{b}{r}\right)\left(1-\frac{\mu}{a}(D+N)\right)\right]\right\}-\lambda I\right| = \begin{vmatrix} A_1^*-\lambda & \mu \\ -B_1^* & 0-\lambda \end{vmatrix}$$

This gives $\lambda^2 - \lambda A_1^* - \mu B_1^* = 0$,

$$\lambda^2 - \left[\left(1-\frac{b}{r}\right)\left(1-\frac{2\mu}{a}(D+N)\right)-\frac{1}{D+N}\left[(D+N)\left(1-\frac{b}{r}\right)\left(1-\frac{\mu}{a}(D+N)\right)\right]\right]\lambda - \mu\frac{1}{D+N}\left[(D+N)\left(1-\frac{b}{r}\right)\left(1-\frac{\mu}{a}(D+N)\right)\right] = 0 \qquad (12)$$

According to the quadratic formula, the solutions are

$$\lambda = \frac{A_1^* \pm \sqrt{(-A_1^*)^2 + 4\mu B_1^*}}{2}$$

The eigenvalues are positive if the value generated by the radical is less than A_1^*, and with all parameter values positive, and the eigenvalues are negative if the value generated by the radical is more than A_1^*. Hence, we have stable equilibrium when the eigenvalues are negative and an unstable equilibrium when the eigenvalues are positive.

This is possible only when conditions (6) and (7) are put in place, that is, $b < r$ and $\mu < \frac{a}{D+N}$. $b < r$, implies that prey's cumulative density b, which is the regulatory term, must be less than its growth rate r, whereas $\mu < \frac{a}{D+N}$, implies that, predator's mortality rate must be less than the quotient of the maximum killing rate over the sum of half-saturation constant and the prey density.

Global stability of the equilibrium points

Definition 1: Positive definite

A function *V(x, y)* which is continuously differentiable is to be positive definite in a region ∪ that contains the origin if

$V(0, 0) = 0$

$V(x) > 0$ and $V(y) > 0$, for all $x, y \in \cup - \{0\}$.

Definition 2: The Lyapunov function

A Lyapunov function *V(x,y)* is defined as follows;

i.) *V* and all its partial derivatives $\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}$ are continuous.

ii.) *V* is positive, that is, $V(0, 0) = 0$ if and only if $x = x_0, y = y_0$ and $V(x) > 0$, $V(y) > 0$, for all $x, y \in \cup - \{0\}$.

A Lyapunov function *V*(*x*, *y*) for a system is said to be

i.) Positive definite if $V(x,y) > 0$ for all $x, y \in \cup - \{0\}$.

ii.) Positive semi-definite if $V(x,y) \geq 0$ for all $x, y \in \cup - \{0\}$.

iii.) Negative definite if $V(x,y) < 0$ for all $x, y \in \cup - \{0\}$.

iv.) Negative semi-definite $V(x,y) \leq 0$ for all $x, y \in \cup - \{0\}$.

Definition 3: Lyapunov stability theorem

Let $(\bar{x},\bar{y})$ be the equilibrium point of the differential equation $\dot{x} = f(x,y)$ and $V(x,y)$ be a continuously differentiable positive definite function in the neighbourhood of the origin.

i.) If $\dot{V}(x,y) \leq 0, \forall\, x,y \in \cup - \{0\}$, then the origin is stable (Lyapunov stable).

ii.) If $\dot{V}(x,y) < 0, \forall\, x,y \in \cup - \{0\}$,then the origin is uniformly asymptotically stable.

If $\dot{V}(x,y) > 0, \forall\, x,y \in \cup - \{0\}$, then the origin is unstable.

Therefore a function $V(x,y)$ is a Lyapunov function if

$V(x,y) = 0$, only if $x = x_0\,, y = y_0$

$V(x,y) > 0, \forall\, x,y \in \cup - \{0\}$

$V(x,y) \leq 0, \forall\, x,y \in \cup - \{0\}$

If $\dot{V}(x,y) < 0$, then $V(x,y)$ is a strict Lyapunov.

To get $\dot{V}(x,y)$, for a two dimensional system, the following formula is used:

$$\frac{dV}{dt} = \dot{V}(x,y) = \frac{\partial V}{\partial x}.\frac{dx}{dt} + \frac{\partial V}{\partial y}.\frac{dy}{dt} = \frac{\partial V}{\partial x}x' + \frac{\partial V}{\partial y}y'$$

It is easy to verify that *V* is zero at the equilibrium and is positive for all values of *x* and *y*, from the definitions stated above.

The non-trivial global stabilities that we are going to look for are those of E_1, E_2 and E_3.

Global stability of $E_1(\bar{N},0) = E_1(1,0)$

We consider a Lyapunov function as $V(N,0) = N - \bar{N} - \bar{N}\, ln\left(\frac{N}{\bar{N}}\right)$.

Differentiating *V* with respect to time *t* we get,

$$\dot{V}(N,0) = \left(\frac{N - \bar{N}}{N}\right)\dot{N}(t)$$

Substituting $\dot{N}(t)$ using equation (4) gives,

$$\dot{V}(N,0) = (N - \bar{N})\left[\left(1 - \frac{b}{r}\right)(1 - N) - \frac{aP}{D + N}\right]$$

For E_1 ($\bar{N}$, 0), $\bar{P} = 0$. Thus,

$$\dot{V}(N,0) = (N - \bar{N})\left[\left(1 - \frac{b}{r}\right)(1 - N)\right]$$

From this we get,

$$\dot{V}(N,P) = (N - \bar{N})\left[\left(\frac{b}{r} + \bar{N} + \frac{b\bar{N}}{r}\right) - \left(\frac{b}{r} + N + \frac{bN}{r}\right)\right]$$

This simplifies to, $\dot{V}(N,P) = -(N - \bar{N})\left[1 + \frac{b}{r}\right]$.

Hence $E_1(\bar{N},0)$ is globally asymptotically stable.

Global stability of $E_2(0,\bar{P}) = E_2(0,0)$

We use the Lyapunov function $V(0,P) = P - \bar{P} - \bar{P}\, ln\left(\frac{P}{\bar{P}}\right)$.

Differentiating *V* with respect to time *t* we get, $\dot{V}(0,P) = \left(\frac{P-\bar{P}}{P}\right)\dot{P}(t)$.

Substituting $\dot{N}(t)$ and $\dot{P}(t)$ using equation (6) gives,

$$\dot{V}(0,P) = \left(\frac{P - \bar{P}}{P}\right)\dot{P}(t)$$

From this we get,

$$\dot{V}(0,P) = (P - \bar{P})\left[\frac{a\bar{N}}{D + \bar{N}} - \frac{aN}{D + N}\right]$$

This simplifies to,

$$\dot{V}(0,P) = -(N - \bar{N})(P - \bar{P})\left[\frac{a}{(D + \bar{N})(D + N)}\right]$$

Hence $E_2(0,\bar{P})$is Lyapunov stable.

Global stability of E_3 ($\bar{N}$,$\bar{P}$)

Consider the following Lyapunov function,

$$V(N,P) = N - \bar{N} - \bar{N}\, ln\left(\frac{N}{\bar{N}}\right) + P - \bar{P} - \bar{P}\, ln\left(\frac{P}{\bar{P}}\right)$$

Differentiating *V* with respect to time *t*, we get,

$$\dot{V}(N,P) = \left(\frac{N - \bar{N}}{N}\right)\dot{N}(t) + \left(\frac{P - \bar{P}}{P}\right)\dot{P}(t)$$

Substituting in the expression for $\dot{N}(t)$ and $\dot{P}(t)$ from equation (4), we get

$$\dot{V}(N,P) = (N - \bar{N})\left[\left(1 - \frac{b}{r}\right)(1 - N) - \frac{aP}{D + N}\right] + (P - \bar{P})\left[\mu - \frac{aN}{D + N}\right]$$

From this we get,

$$\dot{V}(N,P) = (N - \bar{N})\left[\left(\frac{b}{r} + \bar{N} + \frac{b\bar{N}}{r} + \frac{a\bar{P}}{D + \bar{N}}\right) - \left(\frac{b}{r} + N + \frac{bN}{r} + \frac{aP}{D + N}\right)\right]$$

$$+(P - \bar{P})\left[\frac{a\bar{N}}{D + \bar{N}} - \frac{aN}{D + N}\right]$$

This simplifies to,

$$\dot{V}(N,P) = -(N - \bar{N})\left[1 + \frac{b}{r}\right] - (N - \bar{N})(P - \bar{P})\left[\frac{a}{(D+\bar{N})(D+N)}\right]$$

$$-(P - \bar{P})\left[\frac{a}{(D + \bar{N})(D + N)}\right]$$

Therefore, $E_3(\bar{N},\bar{P})$ is globally asymptotically stable.

In this chapter, we looked at the existence of all the possible three equilibrium points. At each point, we identified conditions necessary for them to exist. It was found out that cereal aphids can exist on their own in the absence of ladybirds. However, their population would increase until it reaches the host plants carrying capacity *K*, then start to reduce all over again resulting into a periodic nature of population change. The ladybird cannot exist on their own in the absence of aphids, which they feed on. The co-existence of these two species requires $b < r$ and $\mu h < \frac{a}{D+N}$. These inequalities show the parameters that must be controlled for this co-existence to occur.

The conditions for local and global stability were also established. The conditions for the local stability were in most cases found to be similar to those for the existence. The conditions for the global stability states E_1, E_2 and E_3 were established by developing a suitable Lyapunov function. With the differentiated Lyapunov function, it was found that E_1 and E_3 each gave a negative value, hence were globally asymptotically stable. E_2 was Lyapunov stable. The conditions necessary for the global stability state of E_3 are similar to those for existence. For the global stability of the co-existence of the prey and the predator, the prey's death rate must be less than its growth rate. Also, the predator's mortality rate must be less than the quotient of the maximum killing rate over half the sum of the saturation rate and the population of the prey.

Conclusions

The damage aphids' cause to the crops as well as loss of yields can be extensive if not contained. However, to contain these pests, it is important to understand its dynamics in relation to its interaction with its natural enemies like the ladybird. In this study, we have constructed sets of mathematical models, by adjusting the function representing the prey-predator interaction.

Conflicts of interest

Authors declare no conflict of interest.

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