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Research Article

Derivation Properties of Finite Rank Operators

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Abstract

In the present work, authors established derivation properties and range-kernel orthogonality of finite rank inner derivations implemented by finite rank hyponormal operators. The results show that an inner derivation is linear and bounded. Also by inner product trace and properties of adjoint, the inner derivation is self-adjoint if the inducing operator is self-adjoint. For orthogonality, we employ operator techniques such as properties of operators and derivation inequalities due to Anderson, Bouali, Maher, Mecheri and Halmos generalization formula to establish the orthogonality.

Keywords: Orthogonality; Hyponormal operators; Commutator; Finite rank and derivation.

Introduction

As an area of research, derivation properties and range-kernel orthogonality has attracted many mathematicians. For instance [1] introduced the notion of orthogonality in Banach spaces which generalizes the usual orthogonality in Hilbert spaces while as [2, 3] established operator norm for inner derivations. In [4] they characterized inner derivation with orthogonality and for normal operators established orthogonality inequality; $\|AX - XA + T\| \geq \|T\|$ for all $X \in B(H)$ which implies range-kernel orthogonality for inner derivation. Here orthogonality is defined in [5] sense where $x \in H$ is said to be B-orthogonal to $y \in H$ if $\|x + \lambda y\| \geq \|x\|$. In normed spaces [6, 7, 8] established that Birkhoff orthogonality implies best approximation and best best approximation implies Birkhoff orthogonality and thus the significance of this sense of orthogonality [9,10].

With respect to von Neumann Schatten p-class, C_p, [11] has established range-kernel orthogonality inequality; $\|AX - XA + T\|_p \geq \|T\|_p$ for all $X \in B(H)$ and for a normal operator $A \in B(H)$ which commutes with $T \in C_p$. In the context of the structure of a compact vector space and Hilbert-Schmidt norm, Range properties of derivation have been established by [12]. Furthermore it has been established that a derivation in B(H) is also an inner derivation. This is the case only in finite dimension where the properties of adjoint operators are also inherited by inner derivation. However an inner derivation is also a derivation in general [13]. Operators in $\overline{R(\delta_A)}$ are significant and they have been used by [14] to establish operators in $\overline{R(\delta_A)} \cap \{A\}^{\#}$ to be nilpotent if P(A) is normal, isometric or co-isometric for some polynomial P. Also if $A \in B(H)$ is subnormal and has a cyclic vector or if is isometric, then $\overline{R(\delta_A)} \cap \{A^*\}^{\#} = \{0\}$.

Hyponormal operator is a generalization of many other classes of operators such as finite, normal and log-hyponormal operators. Range-kernel orthogonality conditions for such large class of operators have been established by minimization procedures. For instance for hyponormal operator $A \in B(H)$ such that AT = TA where T is an isometry, then in [15] we have $\|T + AX - XA\| \geq \|T\|$ for all $X \in B(H)$. The inequality still holds if X^* is hyponormal operator which commutes with an isometric operator, T. By using the power norm equality and by compactness properties of hyponormal operators, [16] established approximation results for paranormal operators which in turn has been used to establish orthogonality.

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To establish the required result we use orthogonality inequalities due to [17, 18, 19] and in [20] for normal operators, orthogonal decomposition, algebraic direct sum of operators, algebraic properties of projections and adjoint operators, matrix decomposition of operators, computational skills and techniques to establish range-kernel orthogonality inequalities for finite rank hyponormal operators. We take $F_H^{(H)}$ to denote the algebra of all finite rank hyponormal operators acting on an infinite dimensional Hilbert space H, $R(\delta_A)$ to be range of inner derivation and its corresponding closure as $\overline{R(\delta_A)}$, $\{A\}^{\#}$ the commutator of A ϵ $F_H^{(H)}$ and $\ker\delta_A$ the kernel of inner derivation [21].

Research methodology

Preliminaries

In this section, we start by defining some key terms that are used in the present work.

Definition 2.1 ([13], Definition 1.2.26) The rank of operator A is the dimension of its range. A finite rank operator is a bounded linear operator between Banach spaces whose range is of finite dimension.

Definition 2.2 ([10], Definition 2.1) Orthogonalities:

Let x, y $\in$ H be vectors then; (i). x is orthogonal to y written as $x \perp y$, if $\langle x,y\rangle = 0$

(ii). x is Birkhoff orthogonal to y denoted as $x \perp_B y$ if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{C}$.

(iii). x is Roberts orthogonal to y denoted as $x \perp_R y$ if $\|x + \lambda y\| = \|x - \lambda y\|$ for all $\lambda \in \mathbb{C}$.

iv). x is isosceles orthogonal to y denoted as $x \perp_i y$ if $\|x + y\| = \|x - y\|$

v). x is James orthogonal to y denoted as $x \perp_j y$ if $\|y + \lambda x\| \geq \|x\|$ for all $\lambda \in \mathbb{C}$.

vi). x is Singer orthogonal to y denoted as $x \perp_j y$ if x = 0 and y = 0.

Definition 2.3 ([1], Definition 1.3.3) The orthogonal complement $A^{\perp}$ of a subset A is the set of vectors orthogonal to A i.e $A^{\perp}$ = x $\in$ H : $x \perp y$ for all y $\in$ A. Subsets A and B of H are orthogonal written as $A \perp B$ if $x \perp y$ for every x $\in$ A and y $\in$ B.

Definition 2.4 ([18], Theorem 4) Let B(H) be the algebra of all bounded linear operators acting on Hilbert space H. The mapping δ_A : B(H) $\longrightarrow$ B(H) is called an inner derivation defined as $\delta_A(X) = AX - XA$.

Definition 2.5 ([15], Definition 4.5) Let H be a Hilbert space and B(H) be equipped with the operator norm. The operator δ_A defined on the Banach space B(H) is equipped with the operator norm $\|\delta_A X\| = \sup\{\|\delta_A X\| : \|X\| = 1\}$ for all X ϵ B(H).

Definition 2.6 ([4], Section 2) Let T $\in$ B(H) be compact. Then $s_1(T) \geq s_2(T) \geq \ldots \geq 0$ are the singular values of T i.e the eigenvalues of $\|T\| = (T^*T)^{\frac{1}{2}}$ counted according to multiplicity and arranged in descending order. For $1 \leq p \leq \infty$, $C_p = C_p(H)$ is the set of those compact T $\in$ B(H) with finite p-norm, $\|T\|_p = \left(\sum_{i=1}^{\infty} s_i(T)^p\right)^{\frac{1}{p}} = (tr|T|^p)^{\frac{1}{p}} < \infty$.

Definition 2.7 ([17], Definition 12.11) If T is an operator on Hilbert space H and T^* is the respective adjoint then:

i) T is normal if $TT^* = T^*T$

ii) T is self adjoint or Hermitian if $T = T^*$

iii) T is unitary if $TT^* = I = T^*T$

iv) T is idempotent if $T^2 = T$

v) T is nilpotent if $T^n = 0$ for all n ϵ $\mathbb{N}$

vi) T is a projection if $T^2 = T$ and $T^* = T$

vii) T is binormal if $(T^*T)(TT^*) = (TT^*)(T^*T)$

viii) T is hyponormal if $TT^* \leq T^*T$

ix) T is semi-normal if $TT^* \leq T^*T$ or $TT^* \geq T^*T$ i.e. either T or T^* is hyponormal.

x) T is quasinormal if it commutes with T^*T i.e. $T(T^*T) = (T^*T)T$

xi) T is normaloid if $\|T\| = r(T)$ where r(T) is the specral radius of T.

Definition 2.8 ([9], Problem 127) An isometry is a linear transformation T such that $\|Tx\| = \|x\|$ for all x ϵ H. An isometry is a distance preserving transformation such that if

$\|Tx - Ty\|$ = if $\|x - y\|$ for all x and y.

Definition 2.9 ([9], Problem 134) The polar decomposition of A ϵ B(H) is defined by A = UP, where U is a partial isometry, P is a positive operator and kerU = kerP.

Definition 2.10 ([19], Definition 5.3) If H = M $\oplus$ N, then a projection is a linear map P :

$H \longrightarrow H$ taking X onto M along N defined by $Px = y$, where $x = y + z$ with $y \in M$ and $z \in N$ such that $M = \text{ran}P$ and $N = \ker P$.

Results and discussion

In this section, we give the main results. First we establish properties of derivations implemented by finite rank hyponormal operators and then we establish range-kernel orthogonality of finite rank inner derivations implemented by hyponormal operators.

Proposition 3.1 Let $\delta_A : F_H(H) \rightarrow F_H(H)$ defined by $\delta_A(X) = AX - XA$ be of finite rank.

Then δ_A is a derivation and $\delta_A = \delta_B$ if and only if $B = A - \lambda I$ for all $\lambda \in \mathbb{C}$ and $B \in F_H(H)$.

Proof. From the definition we have; $\delta_A(XY) = AXY - XYA$ (1)

$\delta_B(XY) = BXY - XYB$ (2)

Subtracting (2) from (1) we have; $\delta_A(XY) - \delta_B(XY) = AXY - BXY - XYA + XYB$

$\Longrightarrow (\delta_A - \delta_B)(XY) = (A - B)XY - XY(A - B)$

$\Longrightarrow (\delta_A - \delta_B)(XY) = \delta_{A-B}(XY)$

$\Longrightarrow \delta_A - \delta_B = \delta_{A-B}$ which is a derivation according to [20]

According to [16] the converse is true i.e. if δ is a derivation in $F_H^{(H)}$ then there exist $A \in F_H^{(H)}$ such that $\delta = \delta_A$ ([16], Proposition1.4.4).

For the second part, suppose for $A, B \in F_H^{(H)}$ we have $\delta_A = \delta_B$ then this implies $\delta_A - \delta_B = \delta_{A-B} = 0$. Hence for all $X \in F_H^{(H)}$ we have; $\delta_{A-B}(X) = (A - B)X - X(A - B) = 0$

$\Longrightarrow (A - B)X = X(A - B)$. Setting $A - B = C$ we have $CX = XC$ implying $C = \lambda I$ [16], thus $A - B = \lambda I \Longrightarrow B = A - \lambda I$.

On the other hand, if $B = A - \lambda I$ then by applying derivation on both sides we have;

$\delta_B(X) = \delta_{A-\lambda I}(X)$

$\Longrightarrow BX - XB = (A - \lambda I)X - X(A - \lambda I)$

$\Longrightarrow BX - XB = AX - \lambda X - XA + X\lambda$

$\Longrightarrow BX - XB = AX - XA$

$\Longrightarrow \delta_B = \delta_A$.

Proposition 3.2 Let $\delta_A : F_H(H) \rightarrow F_H(H)$ defined by $\delta_A(X) = AX - XA$ be of finite rank. Then $\delta_A(X) = AX - XA$ is linear and bounded.

Proof. For linearity, let $X, Y \in F_H^{(H)}$ then for scalars $\alpha, \beta \in \mathbb{C}$ we have;

$\delta_A(\alpha X + \beta Y) = A(\alpha X + \beta Y) - (\alpha X + \beta Y)A = \alpha AX - \alpha XA + \beta AY - \beta YA$

$= \alpha(AX - XA) + \beta(AY - YA)$

$= \alpha\delta_A(X) + \beta\delta_A(Y)$. Hence δ_A is linear.

By [8] a derivation is a linear map $\delta : F_H(H) \rightarrow F_H(H)$ satisfying the Leibniz rule; $\delta(XY) = \delta(X)Y + X\delta(Y)$ and by [16] if $\delta : F_H(H) \rightarrow F_H(H)$ is a derivation, then there exist $A \in F_H(H)$ such that $\delta = \delta_A$.

Thus $\delta_A(XY) = \delta_A(X)Y + X\delta_A(Y)$

$\Longrightarrow \delta_A(XY) = (AX - XA)Y + X(AY - YA)$.

But A is finite implying existence of $I \in F_H(H)$ such that $\|AX - XA - I\| \geq I$ and $\|AY - XY - I\| \geq I$ and hence from line (1) we have;

$\|\delta_A(XY)\| \leq \|AX - XA - I\|\|Y\| + \|X\|\|AY - YA - I\|$.

$\Longrightarrow \|\delta_A(XY)\| \leq \|Y\| + \|X\|$.

Thus there exist a positive integer $n \in \mathbb{N}$ such that $\|\delta_A(XY)\| \leq n$.

Proposition 3.3 Let $\delta_A : F_H(H) \rightarrow F_H(H)$ defined by $\delta_A(X) = AX - XA$ be of finite rank. Then $(\delta_A)^* = \delta_{A^*}$ and $R(\delta_A) = [\{A\}^{\#}]^{\perp}$ if and only if $A = A^*$.

Proof. We use definition of inner product trace; $\langle A, B\rangle = \text{tr}(AB^*)$ then for all $X, Y \in F_H^{(H)}$ we have; $\langle(\delta_A)^*X, Y\rangle = \langle X, \delta_A Y\rangle$

$= \text{tr}\{X(AY - YA)^*\}$

$= \text{tr}\{XY^*A^* - XA^*Y^*\}$ (1)

Also $\langle\delta_{A^*}X, Y\rangle = \langle A^*X - XA^*, Y\rangle = \text{tr}\{A^*XY^* - XA^*Y^*\}$ (2)

By definition trace is sum of entries on main diagonal of matrices, hence line (1) is equal to line (2) and indeed; $\text{tr}\{XY^*A^* - XA^*Y^*\} = \text{tr}\{A^*XY^* - XA^*Y^*\} = 0$

$\Longrightarrow (\delta_A)^* = \delta_{A^*}$ (3)

On the second part we use properties of adjoint operator.

Suppose $A = A^*$ then for an arbitrary $x \in R(\delta_A)$ there is a $y \in F_H(H)$ such that $x = \delta_A(y)$.

Now for any $z \in \ker(\delta_A)^*$ and by the previous result we have;

$\langle x, z\rangle = \langle\delta_A(y), z\rangle = \langle y, (\delta_A)^*z\rangle = \langle y, \delta_{A^*}z\rangle = \langle y, \delta_A z\rangle = 0$.

This implies $x \in \{\ker(\delta_{A^*})\}^{\perp}$.

But $x \in R(\delta_A)$ is arbitrary, implying $R(\delta_A) \subset \{\ker(\delta_{A^*})\}^{\perp}$ (4)

On the other hand, let $x \in \{R(\delta_A)\}^{\perp}$, then for all $y \in F_H(H)$ we have;

$0 = \langle \delta_A(y), x\rangle = \langle y, \delta_{A^*}x\rangle \implies (\delta_{A^*})x = (\delta_A)x = 0.$

Hence $x \in \{\ker(\delta_{A^*})\} \implies \{R(\delta_A)\}^{\perp} \subset \{\ker(\delta_{A^*})\}$

By taking orthogonal complement on both sides we have;

$\{\ker(\delta_{A^*})\}^{\perp} \subset \{R(\delta_A)\}^{\perp\perp} = \overline{R(\delta_A)} = R(\delta_A) \implies \{\ker(\delta_{A^*})\}^{\perp} \subset R(\delta_A)$ (5)

Comparing (4) and (5) we have $R(\delta_A) = \{\ker(\delta_{A^*})\}^{\perp}$ (6)

But also $\ker(\delta_A)^* = \{A^*\}^\# = \{A\}^\#$. Then from (6) we have $R(\delta_A) = [\{A\}^\#]^{\perp}$

Suppose $R(\delta_A) = [\{A\}^\#]^{\perp}$ then we show that $A = A^*$.

But also from [16] we have; $R(\delta_A) = \{\ker(\delta_{A^*})\}^{\perp}$

Taking orthogonal complement both sides we have; $\{R(\delta_A)\}^{\perp} = \ker(\delta_A)^*$

Taking orthogonal complement both sides again we have; $\{R(\delta_A)\}^{\perp\perp} = \{\ker(\delta_A)^*\}^{\perp}$

$\implies \overline{R(\delta_A)} = \{\ker(\delta_A)^*\}^{\perp}$

Then $H = \overline{R(\delta_A)} \oplus \{\ker(\delta_A)^*\}^{\perp}$ [17]

$\implies \delta_A$ is a projection and thus self adjoint [19]

$\implies (\delta_A)^* = \delta_A$ (8)

But also from line (3) we have $(\delta_A)^* = \delta_{A^*}$ (7)

From (7) and (8) we have $\delta_A = \delta_{A^*} \implies A = A^*$

Lemma 3.4 Let $A, X \in F_H(H)$, $T \in \{A\}^\#$ and $\alpha > 0$. If A is contractive and if $\|AX - XA - T\| < \alpha$ then $\|(A^{n+1}X - XA^{n+1}) - (n+1)A^nT\| < (n+1)\alpha$.

Proof. By [9] generalization formula for derivation where C is substituted by AX – XA we have;

$A^nX - XA^n - nA^{n-1}T = \sum_{i=1}^{n} A^{n-i-1}((AX - XA) - T)A^i$ (1)

For n =1 we have; $AX - XA - T = AX - XA - T$

For n = 2 we have; $A^2X - XA^2 - 2AT = ((AX - XA) - T)A + ((AX - XA) - T)A$

$\implies A^2X - XA^2 - 2AT = 2((AX - XA) - T)A$

$\implies \|A^2X - XA^2 - 2AT\| \leq 2\|(AX - XA) - T\|\|A\| = 2\alpha$

Similarly for n = 3 we have; $\implies A^3X - XA^3 - 3A^2T = A((AX - XA) - T)A + ((AX - XA) - T)A^2 + ((AX - XA) - T)A^2$

$\|A^3X - XA^3 - 3A^2T\| \leq 3\|(AX - XA) - T\|\|A^2\| = 3\alpha$

For an arbitrary $n \in \mathbb{N}$ we have; $\|(A^nX - XA^n) - nA^{n-1}T\| = n\alpha$

Hence for n + 1 we have $\|(A^{n+1}X - XA^{n+1}) - (n+1)A^nT\| = (n+1)\alpha$.

Lemma 3.5 For $\alpha > 0$, let $A, B \in F_H(H)$, be hyponormal operators such that A is contractive. Suppose there exist $T \in F_H(H)$ such that $AT = TB$ and $\|AX - XB - T\| < \alpha$ then for every $n \in \mathbb{N}$ and for all $X \in F_H(H)$ we have $\|(A^{n+1}X - XB^{n+1}) - (n+1)A^nT\| < (n+1)\alpha$.

Proof. Equality (1) above can be written as $A^nX - XB^n - nA^{n-1}T = \sum_{i=1}^{n} A^{n-i-1}((AX - XB) - T)A^i$

For n = 1 we have; $AX - XB - T = AX - XB - T$

For n = 2 we have; $A^2X - XB^2 - 2AT = ((AX - XB) - T)A + ((AX - XB) - T)A$

$\implies \|A^2X - XB^2 - 2AT\| \leq 2\|(AX - XB) - T\|\|A\| = 2\alpha$

Similarly for n = 3 we have; $\|A^3X - XB^3 - 3A^2T\| \leq 3\alpha$

For arbitrary $n \in \mathbb{N}$ we have; $\|A^nX - XB^n - nA^{n-1}T\| \leq n\alpha$

Hence for n + 1 we have; $\|A^{n+1}X - XB^{n+1} - (n+1)A^{n-1}T\| \leq (n+1)\alpha$.

Lemma 3.6 Let $A \in F_H(H)$, then the following are equivalent;

(i) $I \in \overline{R(\delta_A)}$

(ii) There exist $T \in \{A\}^\#$ such that $T \in \overline{R(\delta_A)}$

(iii) $\overline{R(\delta_A)}$ contains all positive invertible hyponormal operators in $\{A\}^\#$

(iv) $\overline{R(\delta_A)} = F_H(H)$.

Proof. i) $\implies$ ii) Suppose $I \in \overline{R(\delta_A)}$ then also $I \in \{A\}^\#$ implying existence of an invertible operator $T \in F_H(H)$ such that $TT^{-1} = T^{-1}T = I \in \{A\}^\#$. Then by lemma 3.3 in [4] we have a polynomial P of degree n such that $P^k(A)X_n - X_nP^k(A) \longrightarrow P^{k+1}(A)I$

where P^k is the k^{th} derivative of P and (X_n) is a sequence of operators of $F_H(H)$

$\Rightarrow P^k(A)X_n - X_nP^k(A) \rightarrow P^{k+1}(A)TT^{-1}$

Multiplying each term from the right by T we have

$P^k(A)X_nT - X_nP^k(A)T \rightarrow P^{k+1}(A)TT^{-1}T$

By polynomial properties we have, $P(A)X_nT - X_nP(A)T \rightarrow P^1(A)T$.

$\Rightarrow P(A)X_nT - TX_nP(A) \rightarrow P^1(A)T$

$\Rightarrow T \in \{A\}^\#$.

Also $I \in \overline{R(\delta_A)}$ implying existence of a sequence of operators (X_n) such that $AX_n - X_nA \rightarrow I$ and since A is finite hyponormal operator we have $\|AX_n - X_nA - I\| \geq \|I\|$ implying existence of an invertible operator $T \in F_H(H)$ such that $\|AX_n - X_nA - TT^{-1}\| \geq \|TT^{-1}\|$

$\Rightarrow \|AX_n - X_nA - T\|\|T^{-1}\| \geq \|T\|\|T^{-1}\|$
$\Rightarrow \|AX_n - X_nA - T\| \geq \|T\|$
$\Rightarrow T \in \overline{R(\delta_A)}$.
ii) $\Longrightarrow$ i) Suppose there exist an operator P such that $P \in \overline{R(\delta_A)} \cap \{A\}^{\#}$. Then there exist a sequence of operators $\{X_n\}$ of $F_H^{(H)}$ such that $\|P - (AX_n - X_nA)\| \longrightarrow 0$ as setting $n \longrightarrow 0$.
Setting $T_n = P^{-1}X_n$ we have; $\|P^{-1}P - P^{-1}(AX_n - X_nA)\| = \|I - (AP^{-1}X_n - P^{-1}X_nA)\| = \|I - (AT_n - T_nA)\|$ and since $P \in \{A\}^{\#}$ implies that $P^{-1} \in \{A\}^{\#}$ we have; $\|I - (AT_n - T_nA)\| = \|I - (P^{-1}AX_n - P^{-1}X_nA)\|$
$= \|P^{-1}(P - (AX_n - X_nA))\| \leq \|P^{-1}\|\|P - (AX_n - X_nA)\|$.
Since $\|P - (AX_n - X_nA)\| \rightarrow 0$ as $n \rightarrow \infty$ it follows that $\|I - (AT_n - T_nA)\| \rightarrow 0$ as $n \longrightarrow \infty$ and
hence $I \in \overline{R(\delta_A)}$.
i) $\Longrightarrow$ iii) If $I \in \overline{R(\delta_A)}$, then there exist a sequence (X_n) of operators of F_H (H) such that $\|I - (AX_n - X_nA)\| \rightarrow 0$ as $n \longrightarrow \infty$ and also since $I \in \overline{R(\delta_A)}$ then for every invertible operator $B \in \overline{R(\delta_A)}$ there exists $B^{-1} \in \overline{R(\delta_A)}$ such that $I = BB^{-1}$ and hence
$\|I - (AX_n - X_nA)\| = \|BB^{-1} - (AX_n - X_nA)\| \leq \|B^{-1}\|\|B - (AX_n - X_nA)\|$
$\Longrightarrow \|B - (AX_n - X_nA)\| \longrightarrow 0$ as $n \longrightarrow \infty$ which implies that $(AX_n - X_nA) \longrightarrow B \in \overline{R(\delta_A)}$, by definition and by setting $(X_n) \longrightarrow B$ as $n \longrightarrow \infty$ we have $BA = AB$ for an arbitrary positive invertible hyponormal operator $B \in \{A\}^{\#}$.
iii) $\Longrightarrow$ iv) Let $B \in F_H$ (H) then by definition $\{A\}^{\#} = \{B \in F_H (H) : AB = BA\}$ for all $A \in F_H$ (H). Implying that $F_H (H) \subset \{A\}^{\#}$ then by iii) we have $F_H (H) \subset \overline{R(\delta_A)}$ and hence $A \in F_H (H) \subset \overline{R(\delta_A)}$. On the other hand, let $X \in \overline{R(\delta_A)}$ then we need to show that $X \in F_H$ (H). Let $B \in \{A\}^{\#}$ be a positive invertible hyponormal operators such that by iii) we have $B \in \overline{R(\delta_A)}$ then there exist a sequence (X_n) such that $AX_n - X_nA \longrightarrow B \Longrightarrow \|B - (AX_n - X_nA)\| \longrightarrow 0$ as $n \longrightarrow \infty$ and by the vanishing properties of all operators in $\overline{R(\delta_A)}$ we have $AX_n = X_nA$ as $n \longrightarrow \infty$ and by setting $(X_n) \longrightarrow X$ as $n \longrightarrow \infty$ then $AX_n = X_nA$ becomes $AX = XA$ implying that $X \in F_H$ (H) hence $\overline{R(\delta_A)} = F_H$ (H).
At this point, we establish range-kernel orthogonality of finite rank inner derivation implemented by hyponormal operators.

Theorem 3.7

Let $A \in F_H^{(H)}$ be hyponormal operator. Suppose there exist a $T \in F_H^{(H)}$ such that $T \in \{A\}^{\#}$ then for all $X \in F_H^{(H)}$ we have $\|T - (AX - XA)\| \geq \|T\|$ for all $T \in \ker\delta_A$.

Proof. Equality (1) can be written as;
$A^nX - XA^n + \sum_{i=1}^{n} A^{n-i-1}(T - (AX - XA))A^i = nA^{n-1}T$
For n = 1, we have $AX - XA + T - (AX - XA) = T$
$\Longrightarrow \|AX - XA\| + \|T - (AX - XA)\| \geq \|T\|$
For n = 2 we have $A^2X - XA^2 + 2(T - (AX - XA))A = 2AT$
$\Longrightarrow \|A^2X - XA^2\| + 2\|A\|\|T - (AX - XA)\| \geq \|T\|$
$\Longrightarrow \frac{\|A^2X - XA^2\|}{2\|A\|} + \|T - (AX - XA)\| \geq \|T\|$
Similarly for n = 3 we have $\frac{\|A^3X - XA^3\|}{3\|A^2\|} + \|T - (AX - XA)\| \geq \|T\|$
Taking $n \rightarrow \infty$ we have $\|T - (AX - XA)\| \geq \|T\|$.

Theorem 3.8

Let $A \in F_H^{(H)}$ be an invertible finite rank hyponormal operator such that $\|A^n\| = \|A\|^n \leq 1$ where $n \in \mathbb{N}$. If $I \in \overline{R(\delta_A)}$ then for all $X \in F_H^{(H)}$ we have $\|T - (AX - XA)\| \geq \|T\|$ for all $T \in \ker\delta_A$.

Proof. Since $I \in \overline{R(\delta_A)}$ then by Lemma 3.6 there exist $T \in \{A\}^{\#}$ such that the equality
$A^nX - XA^n + \sum_{i=1}^{n} A^{n-i-1}(T - (AX - XA))A^i = nTA^{n-1}$ holds.
We multiply each term from the right by A^{n-1}

$A^nXA^{1-n} - XA^nA^{1-1} + \sum_{i=1}^{n} A^{n-i-1}(T - (AX - XA))A^iA^{1-n} = nTA^{n-1}A^{1-n}$
Which becomes $A^nXA^{1-n} - XA + \sum_{i=1}^{n} A^{n-i-1}(T - (AX - XA))A^{i+1-n} = Nt$
$\Longrightarrow \|A^n\|\|X\|\|A^{1-n}\| + \|X\|\|A\| + \sum_{i=1}^{n} \|A^{n-i-1}\|\|T - (AX - XA)\|\|A^{i+1-n}\| \geq n\|T\|$
$\Longrightarrow 2\|X\| + \sum_{i=1}^{n} \|T - (AX - XA)\| \geq n\|T\|$
$\Longrightarrow \frac{2}{n}\|X\| + \frac{1}{n}\sum_{i=1}^{n} \|T - (AX - XA)\| \geq \|T\|$
$\Longrightarrow \frac{2}{n}\|X\| + \|T - (AX - XA)\| \geq \|T\|$
Taking $n \longrightarrow \infty$ we have $\|T - (AX - XA)\| \geq \|T\|$.

Theorem 3.9 ([2], Theorem 5). Let $T \in F_H^{(H)}$ be hyponormal operator such that $T^n = N$ where n is a positive integer. If N is a normal operator, then T is also normal.

Theorem 3.10 Let $A \in F_H^{(H)}$ be hyponormal operator. Suppose there exist a unitary operator U such that for some positive integer $n \in \mathbb{N}$, $A^n =$

U. Then for all $X \in F_H^{(H)}$ we have $\|T - (AX - XA)\| \geq \|T\|$ for all $T \in \ker\delta_A$.

Proof. By definition U is normal and hence A is also normal by the above theorem.

Let $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ be the matrix representation of A relative to the orthogonal decomposition

$H = H_1 = \overline{R(T)} \oplus \overline{R(T)}^{\perp}$.

Taking matrix representation of X and T on H as $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$ and $T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$.

Then $AX - XA - T = \begin{pmatrix} A_1X_1 - X_1A_1 - T_1 & A_1X_2 - X_2A_2 \\ A_2X_3 - X_3A_1 & A_2X_4 - X_4A_2 \end{pmatrix}$

Since the norm of an operator matrix supersedes/dorminates the norm of its diagonal entry we have;

$\| AX - XA - T\| \geq \|A_1X_1 - X_1A_1 - T_1\| \geq \|T_1\| \geq \|T\|$

$\Longrightarrow \| AX - XA - T\| \geq \|T\|$.

Conclusions

The properties we have established are on inner derivation and the respective range-kernel orthogonality on an algebra of finite rank hyponormal operators acting on an infinite dimensional Hilbert space. We have established and characterized properties of operators in $\overline{R(\delta_A)}$ which forms a very important tool in establishing properties of operators in $\overline{R(\delta_A)} \cap \{A\}^{\#}$ which in turn guarantees range-kernel orthogonality. By polar decomposition and (FP) property it is interesting to investigate hyponormal operators with an aim of establishing orthogonality for hyponormal operators in Schatten p-class.

Conflicts of interest

Authors declare no conflict of interest.

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