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Research Article

On Normality in Dense Topological Subspaces

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Abstract

A lot of studies have been conducted on dense topological spaces over a long period of time and interesting results have been obtained. Normality and compactness on topological spaces have also been investigated for decades however, characterization when the subspaces are particularly dense has not been exhausted. In the present study, we consider the case when the countable subspaces are dense. We introduce the notion of normality in dense topological spaces Also, some characterizations and properties of these notions are investigated.

Keywords: Topological space; Normality; Denseness; Compactness.

Introduction

Many studies have been conducted on dense topological subspaces over a long period of time and interesting results have been obtained [1-4]. Normality and compactness on topological subspaces have also been investigated by many mathematicians [5-7]. However; characterization when the subspaces are particularly dense has not been exhausted. In this study, we consider the case when the countable subspaces are dense. The objective of the study is to characterize normality of dense topological. This work involves a description of both finite and infinite dimensional dense topological subspaces [8-10]. One point compactification and Tychonoff theorems [11-13] have been used in the description of normality and compactness to prove situations where a dense topological subspace is countable. Concerning normality [14-16], results show that a topological subspace X is normal on every dense countable subset.

Moreover, a subspace E of H is strongly normal in H if and only if E is normal in itself and for each continuous real-valued function f on E there exists a real-valued function g on H continuous at all points of E which is an extension of f. Next, we have shown that if continuum hypothesis holds, then there is a countable dense set X of $\mathfrak{R}c$ such that $\mathfrak{R}c$ is normal on X [17-19]. Furthermore, linearly ordered spaces are normal [20-23]. On compactness, it is known that if X is a compact space and Y is a Hausdorff space, then it implies that every continuous bijection f: $X \rightarrow Y$ is a homeomorphism. Also every locally compact Hausdorff space is Tychonoff [24-25].

Research methodology

Definition 1.1

A topological space X is called dense in X if every point x in X either belongs to A or is a limit point of A. That is, the closure of A is constituting the whole set X. A topological space (X, τ) which has countable dense subset is called separable space.

Definition 1.2

A normal space is a topological space in which, for any two disjoint closed sets E and F , there exist two disjoint open sets U and V such that $E \subset U$, and $F \subset V$.

Definition 1.3

A function from A to B is a subset f of A × B such that for all a in A there is exactly one b in B and $(a, b) \in f$. Therefore, we write f: A →

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B for the function f ⊂ A × B and think off as a rule that to any element a ∈ A associates a unique object f (a) ∈ B. The set A is the domain of f and also B is a codomain of f; dom(f) = A, cod(f) = B. The function f is therefore:

Injective or one-to-one if distinct elements of A having distinct images in B, Surjective or onto if all elements in B are images of elements in A, Bijective if both injective and surjective if any elements of B is the image of precisely one element of A.

Definition 1.4

A topological space (X, τ) is called a compact space if every open cover of X has a finite subcover. Compact spaces are always Lindel¨of, that is, they are topological spaces in which every open cover has a countable subcover.

Definition 1.5

Let X be a topological space and let ∞ denote an ideal point called the point of infinity. Let X be a space and ∞ not included in X. So X∞ = X∪∞ is defining a topology ∞ and X∞ by specifying open set:

The open sets of X, considered as subsets of X∞, the subsets of X∞ whose complements are closed, compact subsets of X and, the set X∞.

The space (X∞, τ∞) is called the one point compactification of X.

Definition 1.6

Regular space is a topological space in which every neighborhood of a point contains a closed neighborhood of the same point. While completely regular space or Tychonoff space is a Hausdorff space .

Definition 1.7

Let (X, τ) be a topological space. N be a subset of X and p a point in N . Then N is said to be a neighborhood of the point p if there exists an open set U such that p ∈ U ⊆ N .

Definition 1.8

A family γ of subspaces of subsets of a space X has the finite intersection property provided that every finite sub collection of γ has non empty intersection.

Definition 1.9

Let X be a topological space, Y ⊂ X. We say that Y is internally normal in X if for every two disjoint subsets A and B of Y which are closed in X, There are disjoint sets U and V , open in X, such that A ⊂ U and B ⊂ V . Further, we say that Y is internally compact in X if every M ⊂ Y closed in X is compact.

Definition 1.10

A topological space (X, τ) is called separable if it contains a countable dense subset. There exists a sequence {xn}∞n=1 of elements of the space such that every nonempty open subset of the space contains at least one element of the sequence.

Definition 1.11

A topological space is normal if there exists open sets UA and UB that separate disjoint closed sets A and B. Normality implies that any finite closed number of disjoint closed sets can be simultaneously separated. A convergent sequence with its limit point shows that ”finite” cannot be extended to ”countable”. A space is defined to be collection wise normal if any discrete collection of closed sets can be separated.

Results and discussion

Normal topological spaces are a very important class of topological spaces. We dedicate this section to the study of normality of dense topological subspaces. We state the following proposition.

Proposition 3.1.

For every separable topological sub space X the following conditions are equivalent:

(i). X is normal on every dense countable subset.

(ii). Any two separable disjoint closed subspaces of X can be separated by disjoint open sets.

Proof.

Suppose that axiom (i) holds, we can let sets M and N to be separable disjoint closed sets. Therefore, if we pick a countable set A to be a proper subset of M , that is, A ⊂ M and also, a countable set B, that is, B ⊂ N such that M = A and N = B then since X is separable, there is a countable dense subspace P ⊂ X. Hence, if we

apply conditions for normality of X, we therefore have P ∪ A ∪ B.
Conversely, from [60] it is known that any two separable disjoint closed subspaces of X can be separated by disjoint open sets. This implies that X is normal on every dense countable subset. This is illustrated in the following theorem below.

Theorem 3.2.

Let (Y, τ) be a topological space. Then there exists is a Hausdorff non-regular separable sub space X which is normal on each countable dense subspace.

Proof.

Let Y be a copy of α1 + 1 in ℜc. If we take a countable dense subspace P which exists in ℜc and is also disjoint from Y. Hence space X will be the set Y ∪ P with the following topology:
(i). All points in X, except α1 having their basic neighborhoods that are inherited from ℜc.
(ii). The basic neighborhoods of α1 are the sets, (G ∩ P) ∪ {α1} where G is open set in ℜc that is containing the point α1.
We can therefore easily see that X is a non-regular space. It therefore suffices that X is Hausdorff because its topology is finer than the topology φ on Y ∪ P inherited from ℜc. We claim that the closure of the countable set A in X coincides with φ-closure of A in Y ∪ P . It is true that these closures can differ only in the point α1. Now if α1 ∈ clℜc (A) and, α1 ∈ A, then α1 ∈ clℜc (A \ Y), which can take place if and only if α1 ∈ clX(A). Therefore, from Proposition 4.1, it suffices to prove that any separable disjoint closed sets M and N in X can indeed be separated by open sets. From the above observation, we can see that M and N are closed in (Y ∪ P, φ) which is δ-compact and hence normal. Consequently the sets M and N can be separated in (Y ∪ P, φ) and hence in X.

The following consequences follow immediately.

Corollary 3.3.

Let (Y, τ) be a topological space there exists a Tychonoff separable subspace X which is not normal on any countable dense sub- space.

Proof.

Let the subspace (X, τ) be a T1-space that is if and only if x, y ∈ X and x ⊭ y. Therefore, by Theorem 3.2 this implies that there exists an open set U that contains x and another open set V that is containing y for x ⊭ y, then x ∈ U, but x ∈/ V. Similarly, y ∈ V but y ∈/ U. Hence, the intersection of U and V is be empty, that is, U ∩ V = ϕ so that they are disjoint. A T1- space is said to be normal if and only if whenever A, B are disjoint-closed subsets of X, then there exists open sets U , V in X, A ⊆ U , and also B ⊆ V such that is U ∩ V ⊭ ϕ. Next we move to a fundamental result on normality involving real-valued function on a strongly normal subspace. We state the results as follows.

Theorem 3.4.

Let (H, τ) be a topological space. A subspace E of H is strongly normal in H if and only if E is normal in itself and for each continuous real-valued function f on E there exists a real-valued function g on H continuous at all points of E which is an extensions off.

Proof.

We suppose that, E is strongly normal in H. It also implies that E is normal. If τ is a topology on set H then its topological space is (H, τ). The family μ − {N: N ⊂ H \ E} ∪ τ is a subspace of the generated topology τ ∗ on the set H. Hence, H can be deduce to be endowed with our new topology τ ∗. Therefore (H, τ ∗) is a new topological space. Since E is a subspace of H, we obtain the space HE and so we have (E, τ ∗) being inherited from (H, τ ∗). Therefore, from strong normality of E in H it follows that space HE is also normal. We can therefore, conclude that E is closed in HE due to the fact that E is a subbase of H. So, H and HE generate the same topology on E.

Every continuous function f : E → ℜ can be therefore be extended to a continuous function g : HE → ℜ. If we take any e ∈ E, and since the family {∈ τ : e ∈ E} is a base of HE, at E, the function g is continuous at e with regard to the original topology τ of space H. Conversely, if we take any two closed disjoint non-empty subsets M and N of E, Then since E is normal, Then by [3] there exits a continuous f : E → ℜ, such that f (M) = {0} and f(N) = {1}. If we extend this function, g: H → ℜ is continuous at each point of E. Then the interiors U and V of the sets $\{h \in H : g(h) < 1\}$ and, $\{h \in H : g(h) > 2\}$ are disjoint open subsets of H such that $M \subset U$ and $N \subset V$.

Proposition 3.5.

Let (X, τ) be a normal subspace in (Y, τ). If X is a regular T1-space and Y is dense then Y is a real normal space.

Proof.

Since X is non-empty then the topological space (X, τ) is densely normal and therefore K-normal [31]. Let X to be a regular T1-space and let A and B be any two non-empty subsets of Y such that the closures of A and B are in X and are disjoint. Since Y is dense in X and X is normal there exists disjoint closed subsets P and H in X such that $A \subset P$ and $B \subset H$. Since X is K-normal, then by Theorem 4.4 there exists a continuous real valued function f on X such that f (P) = {0} and f(H) = {1}. Therefore we consider Y to be strongly normal in X if for every two disjoint non-empty subsets A and B of Y closed in Y there exists a Y - continuous function $f : X \to \Re$. Such that f (A) = {0} and f (B) = {1}. Hence, Y is a real normal space. Next we consider normality with regard to a fundamental hypothesis called continuum hypothesis for countable dense subspaces of a topological subspace (X, τ). We state the result as follows.

Theorem 3.6.

Let (Y, τ) be a real topological space. If the continuum hypothesis holds in Y then there exists a countable dense subspace X of $\Re c$ such that $\Re c$ is normal on X.

Proof.

Consider sets E and F and let $\omega = \{(E\alpha, F\alpha) : 1 \leq \alpha < C\}$ be an enumeration of all disjoint pairs of finite subsets of E. We construct subsets $X\alpha = \{x\alpha : n \in \omega\}$ and $Y\alpha = \{y\alpha : 1 \leq v \leq \alpha\}$ of the space $\Re\alpha$such that the following conditions are satisfied:

(i). $X\alpha$ is dense in $\Re\alpha$,

(ii). $x\beta \mid \gamma = x\gamma$, for all $n \in \omega$ and $1 \leq \gamma < \alpha$,

(iii). $y\beta \mid \gamma = y\gamma$, for all $1 \leq v \leq \gamma < \beta < \alpha$,

(iv). If $1 \leq v \leq \alpha$ and $Cl\Re v\ (\delta v(Ev)) \cap Cl\Re v\ (\delta(Fv)) \neq \phi$, then $yv \in Cl\Re\alpha\ (\delta\alpha(Ev)) \cap Cl\Re\alpha\ (\delta\alpha(Fv))$.

Lastly, we consider a result on normality concerning linearly ordered topological. We show that every linear ordered subspace of a Hausdorff space is normal as shown in the next result.

Theorem 3.7.

Let (X, τ) be a Hausdorff space then linearly ordered sub- spaces of X are normal.

Proof.

We need show that every well-ordered space is normal. Consider the half-open interval (p, q], $p < q$. Let P and Q be two disjoint closed subsets and let P0 denote the smallest subset of X. Suppose neither P nor Q contain P0, for any point $p \in P$ then there exists a point, $xp < p$ such that (xp, p] is disjoint from Q. Similarly, for any point $q \in Q$ there exists a point $xq < q$ such that (xq, q] is disjoint from P. Suppose that $p0 \in P \cup Q$, such that $p0 \in P$. The one-point set {p0} = [p0, p+) is open and closed since X is Hausdorff. Therefore we can find disjoint open sets U and V such that $P - \{p0\} \subset U$ and also $Q \subset V$. Then $P \subset U \cup \{p0\}$ and $Q \subset V - \{p0\}$ where the open sets $U \cup \{p0\}$ and $V - \{p0\}$ are disjoint. Hence, linearly subspaces are normal.

Conclusions

In the present paper, we have studied various notions of normality in dense topological spaces. We have introduced Also, some characterizations and properties of these notions have been investigated. The results obtained are useful in explaining deformations and transformations in three dimensional objects.

Conflicts of interest

Authors declare no conflict of interest.

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