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Research Article

On Characterization of Midpoint Locally Uniformly Rotund Norms in Fréchet Spaces

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Abstract

In the present paper, instead of using Hilbert space as the most rotund Banach space, we pick a Fréchet space as a unique Banach space and characterize midpoint locally uniformly rotund Banach spaces.

Keywords: Fréchet space; Midpoint Locally Uniformly Rotund space; Weakly Midpoint Locally Uniformly Rotund space.

Introduction

Let $(X, \|.\|)$ be a Banach space. The following questions have over 20 years of history. They were first asked in 1989 at the fixed point and applications conference in Marseille, Luminy [1]. The standard method of measuring the rotundity of the unit ball in X is through the modulus of convexity,

$\delta_X : [0,2] \to [0,1]$ of X,

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\right\}$$

and the characteristic of convexity,

$\varepsilon_0(X) = Sup\{\varepsilon : \delta_X(\varepsilon) = 0\}$.

The modulus of convexity has two-dimensional character, meaning that

$$\delta_X(\varepsilon) = \inf\{\delta_E(\varepsilon) : E \subset X, \dim E = 2\}.$$

It is known that the Hilbert space H is the most rotund space among all Banach space X, in the sense that

$$\delta_X(\varepsilon) \leq \delta_H(\varepsilon) = \delta_{E_2}(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$$

where E_2 is the two-dimensional Euclidean space. Now, fix $a \in [0,2)$ and consider the class ε_a of all two-dimensional spaces $(E, \|.\|)$ having $\varepsilon_0(E) = a$.

Which of these spaces is the most rotund? It can be formulated in the following questions: (i). Given $\varepsilon \in [a,2)$, what is $Sup\ \{\delta_E(\varepsilon) : E \in \varepsilon_a\}$? (ii). Is there a space $E_a \in \varepsilon_a : \delta_E(\varepsilon) \leq \delta_{E_a}(\varepsilon)\ \forall\ E \in \varepsilon_a$? (iii). Moreover if the answer to these questions is positive, is such a space E_a in some sense unique? The authors in [3] considered spaces L^P and l^P for $P > 1$ and proved that they are uniformly rotund. The author in [4] later considered the Banach products of l^P type, while the authors in [5] expanded the results in [4], enlarged on the family of uniformly rotund Banach spaces and established analogous results for the notion of rotundity of the norm in a Banach space. In our case, we consider a Fréchet space and characterize a midpoint locally uniformly rotund Fréchet spaces. To do this, the following definitions are required.

Research Methodology

Definition 2.1[6, Definition 1.5]

A Fréchet space is a complete Hausdorff metrizable locally convex topological vector space. A trivial example of a Fréchet space is a Banach space.

Definition 2.2 [5, Definition 1.1]

A Banach space $(X, \|.\|)$ is Rotund(R) if given $x, y \in S_X$ with $x \neq y$, then $\|\frac{x+y}{2}\| < 1$.

Definition 2.3[2, Definition 1]

Let X be a Banach space and let A_0 be a collection of nonempty subsets of $X \backslash \{0\}$. For $x \in S, 0 < \varepsilon \leq 2$ and $A \in A_0$, define

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$\delta(x,\varepsilon,A) = \inf\{1 - \left\|\frac{1}{2}(x+y)\right\| : y \in S, \|x - y\| \geq \varepsilon\}$
And $x - y = \alpha z \; \forall \; z \in A_0$. Then X is said to be LUR_{A_0} if and only if $\delta(x,\varepsilon,A)\varepsilon > 0$ for any $x \in S, 0 < \varepsilon \leq 2$ and $A \in A_0$.

Definition 2.4. [5, Definition 1.2]

Let $(X, \|.\|)$ be a Banach space. $(X, \|.\|)$ is Uniformly Rotund (UR) if given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$$

whenever $\| x - y \| \geq \varepsilon$ and $x, y \in S_X$. The function $\delta : [0,2] \to [0,1]$, defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in S_X, \|x - y\| \geq \varepsilon\right\}$$

is called the Modulus of Rotundity(convexity) of the space X.

Definition 1.5. [2, Definition 8]

A Banach space B is Midpoint Locally Uniformly Rotund (MLUR) if whenever x is in B and $\{x\}$ and $\{y\}$ are sequences in B such that $\|x\| = 1$, $\| x_n\| \to 1$,

$\| y_n \| \to 1$ and $\|2x - (x_n + y_n)\| \to 0$, then $(x_n - y_n) \to 0$.

A Banach space X is Midpoint Locally Uniformly Rotund (MLUR)(weakly midpoint locally uniformly rotund(WMLUR)) if for each $\varepsilon > 0$ and $x \in S_X, \delta(\varepsilon, x) > 0, (\delta(\varepsilon, x, f) > 0)$ for each $f \in S_{X^*}$.

Results and discussion

Proposition 3.1[2, Proposition 2]

For a Banach space X, let A be the collection of all norm closed and bounded nonempty subsets of $X \setminus \{0\}$, let B be the collection of all weakly closed and bounded nonempty subsets of $X\setminus\{0\}$, let C be the collection of all $weak^*$ closed and bounded (equivalently, $weak^*$ compact) nonempty subsets of $X^*\setminus\{0\}$ and let D be the collection of all norm compact subsets of $X \setminus\{0\}$. Then (i) X is LUR if and only if X is LUR_A.

(i) X is WLUR if and only if X is LUR_B.

(ii) X^* is W^*LUR if and only if X^* is LUR_C.

(iii) X is R if and only if X is LUR_D.

This result leads us to the following theorem

Theorem 3.2

For a Fréchet space $(E, \|.\|)$, let ε_1 be the collection of all norm closed and bounded nonempty subsets of $E \setminus\{0\}$, let ε_2 be the collection of all weakly closed and bounded nonempty subsets of $E \setminus\{0\}$, let ε_3 be the collection of all $weak^*$ closed and bounded (equivalently, $weak^*$ compact) nonempty subsets of $E^* \setminus\{0\}$ and let ε_4 be the collection of all norm compact subsets of $E \setminus\{0\}$. Define ε_0 as the collection of all weakly compact nonempty subsets of $E \setminus\{0\}$ and $S(e_0;r) = \{e \in E : d(e, e_0) \leq r\}$. Then

(i) $(E, \| . \|)$ is MLUR if and only if $(E, \| . \|)$ is $MLUR_{\varepsilon_1}$.

(ii) (ii) $(E, \|.\|)$ is WMLUR if and only if $(E,\|.\|)$ is $MLUR_{\varepsilon_2}$.

iii) $(E, \| . \|)$ is R if and only if $(E, \|.\|)$ is $MLUR_{\varepsilon_4}$ and if and only if $(E, \|.\|)$ is $MLUR_{\varepsilon_0}$.

Proof:

By Proposition 3.1, statements (i) and (ii) follows hence trivial. We therefore need to proceed and prove (iii). To do that, we note that $(E,\| .\|)$ is $MLUR_{\varepsilon_0}$ implies $(E, \| .\|)$ is $MLUR_{\varepsilon_4}$ trivially and $(E, \| .\|)$ is $MLUR_{\varepsilon_4}$ implies $(E,\| .\|)$ is R follows. We need to show also that $(E, \| .\|)$ is not R if $(E, \| .\|)$ is not $MLUR_{\varepsilon_0}$. Assuming $(E, \| .\|)$ is not $MLUR_{\varepsilon_0}$, there exists e in $S, 0 < \epsilon < 2$ and $\varepsilon_0^* \in \varepsilon_0$ with $\Gamma(e,\varepsilon,\varepsilon_0^*) = 0$. We then choose arbitrary sequences $\{a_n\}$ and $\{b_n\}$ in S, such that $\|2e - (a_n + b_n)\| \to 0, \|(a_n - b_n) > \epsilon\|$ and $(a_n - b_n) = \alpha_n \varepsilon_n^*$ where $\varepsilon_n^* \in \varepsilon_0^*$. Since $\varepsilon_0^* \in \varepsilon_0$, by passing to subsequences, it may be assumed that $\alpha_n \to \alpha$ where $\alpha \neq 0$ and $\varepsilon_n^* \to \varepsilon^*$ weakly where ε^* is in ε_0^*.

Let $a = e + \frac{1}{2}\alpha\varepsilon^*$ and $b = e - \frac{1}{2}\alpha\varepsilon^*$. Then, since $a_n + b_n \to 2e$ and $a_n - b_n \to \alpha\varepsilon^*$ weakly, it follows that $a_n \to a$ weakly and $b_n \to b$ weakly and hence $\| a\| < 1$ and $\| b\| < 1$. But $\| a + b\| = 2$ and so in fact a and b are in S. We further note that $a - b = \alpha\varepsilon^*$ and $\alpha\varepsilon^* \neq 0$ since ε^* is in ε_0^*.

Theorem 3.3

Let $(F, \|.\|_f)$ be a Fréchet space. Then the norm $\|.\|_f$ is not MLUR.

Proof:

For $f = (f_1^1, f_2^2, \ldots)$ in F, let $f^1 = (0, f^2, \ldots)$ and define the equivalent norm $\| f \|_S = \max\{|f^1|, \| x^1 \|_2\}$. Let $\{\alpha_n\}$ be a sequence of positive real numbers decreasing to zero and define the continuous linear injection $T: F \to F$ by $T(f^1, f^2, \ldots) = (f^1, \alpha_2 f^2, \alpha_3 f^3, \ldots)$. For $f_1 \in F$ define $\| f \|_F = (\| f \|_S^2 + \| Tf \|_2^2)^{\frac{1}{2}}$. Then $\|.\|_f$ is an equivalent norm on F. We claim that the norm $\|.\|_f$ is not MLUR. To see this, let $\alpha = (\frac{1}{\sqrt{2}})$ and let

$x = \alpha e_1, x_n = \alpha(e_1 + e_n)$ and $y_n = \alpha(e_1 - e_n)$. Then $\| x \|_f = 1$, $\| x_n \|_f \to 1$, $\| y_n \|_f \to 1$ and $\| 2x - (x_n - y_n) \|_f \to 0$, but $\| x_n - y_n \|_f \to \sqrt{2}$.

Theorem 3.4

Let (E, $\|.\|$) be a Fréchet space of l_∞ type. If E contains C_0, then E does not admit any equivalent MLUR norm.

Proof:

Let $(|||.|||)$ be an equivalent norm on E. Then this norm is not MLUR. To show this, we let $W_\varepsilon = \{f \in E: \|.\|_\infty = 1,\ P \setminus \mathrm{Sup}(f)\}$ is infinite,

$R_\varepsilon = Sup\{\|| f \||: f \in W_\varepsilon,\quad r_\varepsilon = inf\{\|| f \||: f \in W_\varepsilon\}$ ....3.4.1.

Choose an element f_ε of E such that $| \| f_\varepsilon \| | > \frac{(3R_\varepsilon + r_\varepsilon)}{4}$. Then select two disjoint infinite subsets P_0' and P_1' of $P \setminus \mathrm{Sup}(f_\varepsilon)$ with $\epsilon P_i \in E$ for some $k_i \in P_1'$, we define $P_i = P_1' \setminus \{k_i\}$ and let $W_i = \{f \in W_\varepsilon: f(n) = f_\varepsilon(n)\}$ $\forall n$ not in $P_i, (i = 0,1)$ ... ... 3.4.2. Suppose that some $q \in Q$, with $|q| < n$, W_q is specified then put

$R_q = Sup\{\|| f \||: f \in W_q, r_q = inf\{\|| f \||: f \in W_q\}$ 3.4.3

Then let $f_q \in W_q$ satisfy $| \| f_q \| | > \frac{(3R_q + r_q)}{4}$ and take two disjoint infinite subsets P_{q_0}' and P_{q_1}' of $P_q \setminus \mathrm{Sup}(f_q)$ with $\epsilon P_{q_i}' \in E$, put $P_{q_i} = P_{q_i}' \setminus \{k_{q_i}\}$ and define

$W_{q_i} = \{f \in W_q: f(n) = f_q(n)\}$ $\forall n$ not in $P_{q_i}, (i = 0,1)$ ... ... 3.4.4.

Thus by induction on $|q|$, we obtain a family $\{W_q\}_{q \in Q}$ of subsets of $(E, \|.\|)$, a family $\{f_q\}$ of elements of E, a family of $\{P_q\}$ of infinite subsets of P and a family of integers $\{K_q\}$ with the following properties.

W_{q_i} is of the form

$W_{q_i} = \{f \in W_q: f(n) = f_q(n)\}$ $\forall n$ not in $P_{q_i}, (i = 0,1)\ \forall q \in Q$ ......4.4.5 and $f_q(k_q) = 0\ \forall q \in Q, (i = 0,1)$, where R_q and r_q denote the supremum and infimum of $\{\|| f \||: f \in W_q\}$ respectively.

whenever $q \prec h$ and $P_q \cap P_h = \beta$, if h and q are not comparable.

for $q \prec h$. By (v), $\{\vartheta_q\}_{q \in Q}$ defined by

$\vartheta_\varepsilon = f_\varepsilon$, $\vartheta_{q_i} = f_{q_i} - f_q$,

$(i = 0,1)$ 3.4.6

, is a disjoint family of elements of $(E, \|.\|)$. By the tree completeness of E, there exists some $\epsilon \in \{0,1\}^P$ such that

$$f_\varepsilon(E) = f_\varepsilon + \sum_{n \in \mathbb{N}} \vartheta_{\epsilon|_n} \in E \ldots \ldots \ldots \ldots .3.4.7$$

Let $\{k_\mu(n)\}$ be a subsequence of $\{k_{\epsilon|_n}\}$ such that $\epsilon_H \in E$, where $H = \{k_{\mu(1)}, k_{\mu(2)}, \ldots\}$. Also let $H_n = \{k_{\mu(n)}, k_{\mu(n+1)}, \ldots$ and $y_n = \epsilon_{H_n}$. By (i) and (ii),

$\vartheta_{n+1}^+ = f_\varepsilon + y_{n+1}$ and $\vartheta_{n+1} = f_\varepsilon - y_{n+1}$ are in $W_{\epsilon|_n}$. Next , select some $\delta \in E^*$ such that $\delta(y_1) = 1$ and $\delta(\vartheta) = 0\ \forall \vartheta \in C_0$. For such an element δ and $n \in P$, we have $\delta(y_n) = 1$. By (i), $2f_\varepsilon - f \in W_{\epsilon|_n}$, thus $| \| 2f_{\epsilon|_n} - f \| | \leq R_{\epsilon|_n}\ \forall\ f \in W_{\epsilon|_n}$ and $n \in P$. It follows that

$\frac{(3R_{\epsilon|_{n-1}} + r_{\epsilon|_{n-1}})}{2} \leq | \| 2f_{\epsilon|_n} \| | \leq R_{\epsilon|_n} + \|| f \||, \forall f \in W_{\epsilon|_n}$ ..4.3.8

and so

$\frac{(3R_{\epsilon|_{n-1}} + r_{\epsilon|_{n-1}})}{2} \leq R_{\epsilon|_n} + r_{\epsilon|_n} \leq R_{\epsilon|_{n-1}} + \leq r_{\epsilon|_{n-1}}\ \forall n \in P$ 4.3.9

Therefore

$R_{\epsilon|_n} - r_{\epsilon|_n} \leq R_{\epsilon|_n} - \frac{(R_{\epsilon|_{n-1}} + r_{\epsilon|_{n-1}})}{2} \leq R_{\epsilon|_{n-1}} - \frac{(R_{\epsilon|_{n-1}} + r_{\epsilon|_{n-1}})}{2} = \frac{(R_{\epsilon|_{n-1}} - r_{\epsilon|_{n-1}})}{2}$ 4.3.10

The above relation shows that

$\left| \| \vartheta_{n+1}^\pm \| \right| - \|| f_\varepsilon \|| \leq R_{\epsilon|_n} - r_{\epsilon|_n} \leq \frac{(R_{\epsilon|_{n-1}} - r_{\epsilon|_{n-1}})}{2} \leq \frac{R_\varepsilon - r_\varepsilon}{2^n}$ ……….4.3.11

Hence that $(E, \|.\|)$ does not admit any MLUR norm.

Conclusions

Rotundity of norms in Banach spaces has been studied by various authors over a period of time now. Some of the properties of rotundity that has been in the heart of authors include: Uniformly Rotund, Locally Uniformly Rotund, Midpoint Locally Uniformly Rotund, Weakly Uniformly Rotund, Uniformly Rotund in Every Direction, Highly Rotund among others. We have characterized and shown that Midpoint Locally Uniformly Rotund norms exist in Fréchet space.

Conflicts of interest

Authors declare no conflict of interest.

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