*Research Article*

# Characterizations of Derivations on Prime Rings

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**Abstract**

In the present paper, we have studied a $(\alpha,\alpha)$-symmetric derivations D on semiprime rings and prime rings R, we give some results when R admits a $(\alpha,\alpha)$-symmetric derivations D to satisfy some conditions on R.(i)$D([x,y]n+1) = 0$ for all x, y$\in$ R. (ii) $[D(xn+1),\alpha(y)] = 0$ for all x, y $\in$R. (iii) $[[D(x),\alpha(x)],\alpha(x)]= 0$ for all x $\in$R. Where $\alpha$: R $\rightarrow$R is an automorphism mapping.

**Keywords:** Semiprime Ring; Prime Ring; $(\alpha,\alpha)$-Derivations; $(\alpha,\alpha)$-Symmetric Derivation.

## Introduction

Several researchers always ask why derivation?. Derivations on rings help us to understand rings better and also derivations on rings can tell us about the structure of the rings. For instance a ring is commutative if and only if the only inner derivation on the ring is zero. Also derivations can be helpful for relating a ring with the set of matrices with entries in the ring [1]. Derivations play a significant role in determining whether a ring is commutative, see [2], [3] and [4]. Derivations can also be useful in other fields. For example, derivations play a role in the calculation of the eigenvalues of matrices [5] which is important in mathematics and other sciences, business and engineering. Derivations also are used in quantum physics [7]. A lot of work has been done in this field [8, 9, 10].

In [11] he was introduce the $(\alpha,\alpha)$-derivation and $\alpha$-commuting mapping in the following way: If $[f(x),\alpha(x)] = 0$ for all x $\in$R, then f is said to be $\alpha$-commuting, where $\alpha$ is an automorphism. An additive map D: R$\rightarrow$ R is said to be an $(\alpha,\alpha)$-derivation if $D(xy)=D(x)\alpha(y)+\alpha(x)D(y)$.In [12] the authors studied the properties of Jordan $(\alpha,\alpha)$-derivation. Derivations are also generalized as $\alpha$-derivations, $(\alpha,\beta)$-derivations and have been applied in the solution of some functional equations [13] he was proved, let n $\geq$ 2 be a fixed positive integer

and let R be a non-commutative n!-torsion free prime ring. Suppose that there exists a symmetric derivation $\Delta$: $R^n \rightarrow R$ such that the trace $\delta$ of $\Delta$ is commuting on R. Then we have $\Delta = 0$ [4]. In [5] they proved let N be a 2-torsion free 3-prime near-ring, D a symmetric bi-$(s,\tau)$-derivation of N and d the trace of D. If $xd(N) = 0$ for all x $\in$ N, then x = 0 or D = 0,where a near ring N is 3-prime if $aNb = \{0\}$ implies that a= 0 or b= 0,and a mapping $D:N\times N \rightarrow N$ is said to be symmetric if $D(x,y) =D(y,x)$ for all x,y $\in$N. A mapping d: N $\rightarrow$N denoted by $d(x) =D(x,x)$ is called the trace of D where $D:N\times N\rightarrow N$ is asymmetric mapping.

## Preliminaries

Throughout, R is an associative semiprime ring. We shall write [x,y] for xy − yx. Then $[xy,z]= x[y,z] + [x,z]y$; $[x,yz]=y[x,z]+ [x,y]z$ for all x, y,z $\in$ R. Recall that R is prime if aRb=(0) implies a= 0 or b = 0 and semiprime if aRa=(0) implies a = 0. An additive mapping D from R in to itself is called a derivation if $D(xy) = D(x)y + xD(y)$ for all x, y $\in$R. A mapping f of R into itself is called commuting if [f(x), x]=0,and centralizing if [f(x),x] $\in$Z(R) for all x $\in$ R, Z(R) denotes the centre of R. An additive mapping F from R to R is said to be a commuting (resp. centralizing) if [F(x),x]=0 (resp.[F(x),x] $\in Z(R)$) holds for all *x* $\in$R, and d is

said to be central if F(x) $\in Z(R)$ holds for all *x* $\in$R.

More generally, for a positive integer n, we define a mapping F to be n-commuting if [F(x), $x^n$] = 0 for all *x* $\in$R. An additive (α,α)-derivation D:R → R is called an (α, α)-symmetric derivation on a ring R if D(xy) = D(yx) for all x,y ∈ R. The notion of a reverse derivation as an additive mapping d from a ring R into itself satisfying d(xy) = d(y)x+ yd(x),for all x, y ∈ *R is* known [8]. Obviously, if R is commutative, then both derivation and reverse derivation are the same. An additive map D:R → R is said to be an (α, α)-derivation if D(xy) = D(x)α(y)+α(x)D(y) for all *x, y* ∈ *R*. We shall need the following well-known and frequently used lemmas.

### Lemma 2.1

Let R be a semiprime ring and D derivation of R,U a nonzero left ideal of R and $r_R(U)=0$.If D is centralizing or skew-centralizing on U, then D(R) $\in Z(R)$ and the ideal generated by D(R) is in the Z(R)(see [2]).

### Lemma 2.2

A mapping D on a semiprime ring R is a reverse derivation if and only if it is a central derivation (see [7]).

### Lemma 2.3

Let R a ring and D be the (α, α)-symmetric mapping on a ∈ R, then the following statements (see [5]) hold for all x, y ∈R.
(i) D([x,y]) = D([y,x])= 0
(ii) D(x[x,y])= 0 = D([x,y]x)
(iii) D([x,y])2 = 0.

### Remark 2.4

If R is a ring with unity, Inv(R) is the set of all invertible elements of R, then (Inv(R),.) form a group. Let a ∈ Inv(R) then x → $a^{-1}xa$ is an automorphism, referred as inner automorphism, the collection Inv(R) forms subgroup of Aut(R). We denoted to the right annihilator ideal by $r_R(U)=0$,whereU a nonzero left ideal of R.

## Results and discussion

### Theorem 3.1

Let R be a 2-torsion free ring and D an (α,α)-symmetric derivation and α an automorphism

such that [D(x),α(x)]=0 for all x ∈ R, then D is a reverse derivation.

### Proof

By hypothesis [D(x),α(x)] = 0 for all x ∈R. Linearization gives [D(x),α(y)]=[α(x),D(y)] (1) Then [D(x),α(y)]= [D(y),α(x)] (2). Using (1) and (2), we get [D(y),α(x)]=[α(x),D(y)] implies that D(y)α(x)−α(x)D(y)= 0 this gives [D(y),α(x)] = 0 for all x, y ∈R. Replacing x by $α^{-1}(w)$, we get [D(y),w] = 0 for all w, y ∈R. Hence D is a central mapping. Then according to Lemma 2.2., a mapping Don a semiprime ring R is a reverse derivation.

### Theorem 3.2

Let R is a ring and D is an (α,α)-symmetric mapping on a ring R, then D([x,y]n+1) =0 for all x, y∈R, where n is a positive integer.

### Proof

We can prove the theorem with the help of mathematical induction. (i) When n=1, then we have D([x,y]2) =0, As D is an (α,α)-symmetric mapping on a ring R, so by D is an (α, α)- symmetric mapping on a ring R, we have D([x,y]) = D([y,x]) = 0 for all x, y ∈R. Then D([x,y]) = 0 for all x, y ∈R. Also D([x,y]2)=D([x,y])α([x,y])+α([x,y])D([x,y]).

Since, we have D([x,y])=0 for all x, y ∈R. Therefore, D([x,y]2) = 0. (ii) When n=m, then D([x,y]n+1) =0 for all x ∈ R. (iii) Suppose that true when n=m+1, then D([x,y]m+1) =0 for all x∈R. D([x,y]m [x,y]) =0. Then D([x,y]m)α([x,y])+α([x,y]m)D([x,y])=0.

According to Lemma 2.3.(iii), we get α([x,y]m)D([x,y])=0. Again according D is an (α,α)-symmetric mapping on a ring R, we have D([x,y])m =0 for all x, y ∈R. Then by right multiplying by α([x,y]),we get D([x,y])m α([x,y])=0 for all x, y ∈R. Then from them relations, we get D([x,y])m α([x,y])+ α([x,y]m)D([x,y])=0 for all x, y ∈R. D([x,y])m =0 for all x, y ∈R. Right-multiplying by D([x,y]) for all x, y ∈R, leads to D([x,y])m+1 =0 for all x, y ∈R. This completes the proof.

### Theorem 3.3

Let R is a prime ring and D be a (α,α)-symmetric derivation on R, then either R is commutative or D is zero.

***Proof***

By Lemma2.3 (ii) $D(x[x,y])=0$ implies that $D(x)\alpha([x,y]) +\alpha(x)D([x,y])= 0$ also in view of Lemma2.3(i), we have $D(x)\alpha([x,y]) = 0$ for all $x,y \in R$. Further in view of Lemma2.3 (i), $D([x,y]x)=0$ gives $\alpha([x,y])D(x) = 0$. Replacing y by yz in $\alpha([x,y])D(x) = 0$, and using it again, we get $\alpha([x,y])\alpha(z)D(x) = 0$ (3). Replacing x by x + u in $\alpha([x,y])D(x) = 0$, we get $\alpha([x,y])D(u) = \alpha([y,u])D(x)$ for all x, y, u ∈ R (4) Pre multiplying (4) by $\alpha([x, y])D(u)v$ we get $\alpha([x,y])D(u)v\alpha([x,y])D(u)=\alpha([x,y])D(u)v\alpha([y,u])D(x)$. (5) Replacing z by $\alpha-1(D(u)v\alpha([y,u]))$ in (3) and then using in (5), we get $\alpha([x,y])D(u)v\alpha([x,y])D(u)= 0$ for all x, y, u, v ∈ R. As R is a semiprime, therefore, $\alpha([x,y])D(u) = 0$. Replacing y by yz in $\alpha([x,y])D(u) = 0$, we get $\alpha([x,y])\alpha(z)D(u) = 0$. Replacing z by α−1(w), we get $\alpha([x, y])wD(u) = 0$ for all x, y,w, u ∈R. As R is a prime ring, therefore either $\alpha([x,y])= 0$ or $D(u) = 0$ for all x, y, u ∈R. By other word, we have $D(u) = 0$ for some u ∈R, or $\alpha([x,y]) = 0$. That is $([x,y]) = 0$ for all x, y ∈R and hence R is commutative or D is zero (α,α)- symmetric derivation on R.

***Theorem 3.4***

Let R is a semiprime ring and D be an (α,α)-symmetric derivation on R, then $[D(x^{n+1}), \alpha(y)] = 0$ for all x,y ∈R, where n is a positive integer.

***Proof***

Since D is an (α,α)-symmetric derivation, according to Lemma 2.4, replacing z by xz in $D(x[y,z]) = 0$ for all x,y,z∈R, and then using $D(x[y,z]u) = 0$ for all x,y,z∈R. Again replacing z by xz in the equation obtained and using $D(x[y,z]u) = 0$ for all x,y,z∈R, again, we have $D(x^3[y, z]) = 0$. Continuing this process, we get $D(x^{n+1}[y,z]) = 0$ for all x, y, z ∈ R. As D is an (α,α)-symmetric derivation, therefore $D([y,z]x^{n+1}) = 0$ for all x, y, z ∈ R. $D(x^{n+1}[y,z]) = 0$ for all x, y, z ∈R. In view of Lemma 2.3 (i), we get
$$D(x^{n+1})\alpha([y,z]) = 0. \quad (6)$$
As D be an (α,α)-symmetric derivation, therefore, $D([y,z]x^{n+1}) = 0$. In view of Lemma2.3 (i), we have
$$\alpha([y,z])D(x^{n+1}) = 0. \quad (7)$$
Subtracting (7) from (6), we get $[D(x^{n+1}), \alpha([y,z])] = 0$. Replacing z by yz and used again it, we get $[D(x^{n+1}), \alpha(y)] \alpha([y,z]) = 0$.

***Theorem 3.5***

Let R be a 2-torsion free semiprime ring, U a nonzero left ideal of R such that $rR(U)=0$,and α: R → R is an automorphism. If there exists an (α,α)-derivation D:R → R such that $[[D(x),\alpha(x)],\alpha(x)]= 0$ for all x∈U, then D(R) ∈Z(R) and the ideal generated by D(R) is in the Z(R).

***Proof***

At first, we define a mapping H (.,.): R×R → R by $H(x,y)= [D(x),\alpha(y)]+[D(y),\alpha(x)]$ for all x∈ U, y ∈R. Then it is easy to see that $H(x,y) = H(y,x)$ for all x∈ U, y, z ∈R, and additive in both arguments. By using the Jacobian identities and definition of H(x,y), we can conclude the following:
$$H(xy,z)=H(x,z)\alpha(y)+\alpha(x)H(y,z)+D(x)[\alpha(y),\alpha(z)]+[\alpha(x),\alpha(z)]D(y). \quad (8)$$
Define the mapping h:R→ R such that $h(x)= H(x,x)$, then $h(x)=2[D(x),\alpha(x)]$. (9)
One can conclude $h(x+y) = h(x)+h(y)+2H(x,y)$. By $[[D(x),\alpha(x)],\alpha(x)]= 0$,we have, $[h(x),\alpha(x)]= 0$. Linearization of last equation gives $[h(x),\alpha(y)]+[h(y),\alpha(x)]+2[H(x,y),\alpha(x)]+2[H(x,y),(y)]=0$. (10).

Replacing x by −x in (10) and using the fact $h(−x) = h(x)$, we get $[h(x),\alpha(y)]−[h(y),\alpha(x)]+[2H(x,y),\alpha(x)]−[2H(x,y),\alpha(y)]=0$.(11). Adding (10) and (11) and using the fact that R is 2-torsion free, we get $[h(x),\alpha(y)]+2[H(x,y),\alpha(x)]=0$. (12) replacing z by $z\alpha-1D(x^{n+1})$ and again replacing z by $\alpha-1(w)w[\alpha(y),D(x^{n+1})] = 0$. As R is semiprime, therefore, we in the result obtained, we get $[D(x^{n+1}), \alpha(y)]w[\alpha(y),D(x^{n+1})] = 0$. have $[\alpha(y),D(x^{n+1})] = 0$ implies that $[D(x^{n+1}), \alpha(y)] = 0$ for all x, y ∈ R. This completes the proof.

***Corollary 3.6***

Let D be an (α,α)-symmetric derivation on a semiprime ring R, such that $[D(x^{n+1}),\alpha(y)] = 0$ for all x,y ∈R, where n is a positive integer, then $D(x^{n+1})$ is n+1-α-central (resp. n+1-α-commuting) of R.

***Proof***

Replacing y by xy in (12) and using $H(x,y) = H(y,x)$, we get $[h(x), \alpha(x)\alpha(y)]+2[H(xy,x), \alpha(x)] = 0$. In view of (8) and (9), the last expression becomes
$$\alpha(x)([h(x),\alpha(y)]+2[H(x,y),\alpha(x)])+2h(x)[\alpha(y),\alpha(x)]+2[D(x), \alpha(x)][\alpha(y),\alpha(x)]+2D(x)[[\alpha(y), \alpha(x)],$$

α(x)] =0. In view of (9) and (12), the last equation reduces to 3h(x)[α(y),α(x)]+2D(x)[[α(y),α(x)],α(x)]=0. (13) Replacing y by yx in (12) then using (9), (12) and the fact that [h(x), α(x)] = 0,we get 3[α(y),α(x)]h(x)+2[[α(y),α(x)],α(x)]D(x)=0. (14) Replacing y by yz in (13), we get 3h(x)[α(yz), α(x)]+2D(x)[[α(yz),α(x)]α(x)] = 3h(x)α(y)[α(z), α(x)]+3h(x)[α(y),α(x)]α(z)= 0. In view of (13) the last equation becomes 3h(x)α(y)[α(z),α(x)]+4D(x)[α(y),α(x)][α(z),α(x)] + 2D(x)α(y)[[α(z),α(x)],α(x)]=0. (15) Putting y= α⁻¹(D(x)), z = y and using (9) the last equation becomes: 3h(x)D(x)[α(y),α(x)]+2D(x)h(x)[α(y),α(x)]+2(D(x))2[[α(y),α(x)],α(x)]=0 (16)

Pre multiplying (13) by D(x) and subtracting from (16), we get (3h(x)D(x)−D(x)h(x))[α(y),α(x)]=0. (17). Post multiplying (17) by α(z), we get (3h(x)D(x)−D(x)h(x))[α(y),α(x)]α(z)=0. (18). Replacing y by yz in (17), we get (3h(x)D(x)−D(x)h(x))(α(y)[α(z),α(x)]+[α(y),α(x)]α(z))=0. (19). Subtracting (18) from (19), we get {3h(x)D(x)−D(x)h(x)}α(y)[α(z),α(x)]=0. (20)

Replacing z by α−1(2D(x)) and y by α−1(t) in (20), we get {3h(x)D(x)−D(x)h(x)}th(x)=0 for all x∈ U, y ,t ∈R (21). Multiplying (21) by 3D(x) and again replacing t by tD(x) in (21) and then subtracting, we get {3h(x)D(x)−D(x)h(x)}t{3h(x)D(x)−D(x)h(x)}= (22).

Semiprimeness of R implies that 3h(x)D(x)−D(x)h(x) = 0. That is 3h(x)D(x) =D(x)h(x) (23). Now replacing y by zy in (14) and using (14) again and then putting y = α⁻¹(D(x)) and z = y, we get 3[α(y),α(x)]D(x)h(x)+2[α(y),α(x)]h(x)D(x)+2[[α(y),α(x)],α(x)](D(x))2=0. (24). Post multiplying (14) by D(x) and using in (24), we get [α(y),α(x)](3D(x)h(x)−h(x)D(x))=0 (25). Replacing y by zy in (25) and using (25) again, we get [α(z), α(x)]α(y)(3D(x)h(x)− h(x)D(x)) = 0. Replacing z by α⁻¹(2D(x)) in the last equation and using (9), we get h(x)α(y)(3D(x)h(x)− h(x)D(x)) = 0 (26). Pre multiplying (24) by 3D(x) and replacing y by α⁻¹(t), we get3D(x)h(x)t(3D(x)h(x)−h(x)D(x)) = 0 (27). Replacing y by α−1(D(x)t) in (26), we get h(x)D(x)t(3D(x)h(x)− h(x)D(x)) = 0 (28).

Subtracting (28) from (27) and using the fact that R is semiprime, we get 3D(x)h(x) = h(x)D(x). (29).

Using (29) in (23) and by 2-torsion freeness of R, we get h(x)D(x) = 0 (30) and also D(x)h(x) = 0. Now take h(x)D(y)+2H(x,y)D(x)=h(x)D(y)+2([D(x),α(y)]+ [D(y), α(x)])D(x). Replacing y by x and using (9) and (30), we get h(x)D(y) + 2H(x,y)D(x) = 0 (31). h(x)D(y) = −2H(x,y)D(x) (32). Replacing y by xy in (31), we get h(x)α(x)D(y)+2h(x)α(y)D(x)+2α(x)H(y,x)D(x)+ 2D(x)[α(y), α(x)]D(x) = 0. In view of (32) and (30) the last equation becomes [h(x),α(x)]D(y)+2h(x)α(y)D(x)+2D(x)[α(y),α(x)]D(x) = 0. Using [h(x),α(x)]= 0 and 2-torsion freeness of R the last equation becomes h(x)α(y)D(x)+D(x)[α(y),α(x)]D(x)=0 (33). Replacing y by yx in (33), we have h(x)α(y)α(x)D(x)+D(x)[α(y),α(x)]α(x)D(x)=0 (34). Post multiplying (33) by α(x), we get h(x)α(y)D(x)α(x)+D(x)[α(y),α(x)]D(x)α(x)=0 (35). Subtracting (34) from (35) and using (9), we get h(x)α(y)h(x)+D(x)[α(y),α(x)]h(x)= 0 (36). Replacing y by α⁻¹(t) in (36), we get h(x)th(x) + D(x)[t,α(x)]h(x) = 0 (37).

Replacing y by α⁻¹(t) in (15)and then replacing z by α⁻¹(2D(x)) and using (9), we get 3h(x)th(x)+4D(x)[t,α(x)]h(x)+2D(x)t[h(x),α(x)]= 0. As [h(x),α(x)]=0, therefore the last equation becomes 3h(x)th(x) +4D(x)[t,α(x)]h(x)=0 (38). In view of (37), we get h(x)th(x) = 0 and semiprimeness of R implies that h(x)= 0 for all x ∈R. That is [D(x),α(x)]= 0 for all x ∈R. Since α:R → R is an automorphism, then we obtain [D(x), x] = 0 for all x ∈U. This lead to [D(x),x] ∈Z(R), for all x ∈U. Thus, according to Lemma 2.1,we get D(R) ∈Z(R) and the ideal generated by D(R) is in the Z(R).This completes the proof.

### *Corollary 3.7*

Let R be a 2-torsion free non-commutative semiprime ring and α:R → R is an automorphism. Suppose that there exists an (α,α)-inner derivation D:R → R defined by D(x)= α([a,x]) for all x ∈U such that [[D(x),α(x)],α(x)]= 0 then D is skewinner derivation of R.

### *Proof*

From (30) in Theorem 3.6, with using R is noncommutative semiprime ring, we obtain D(x)=0 for all x ∈U. Left-multiplying by x, we

obtain xD(x)=0for all x ∈U. Again right multiplying by x, we get D(x)x=0for all x ∈U. From these equation, we obtain D(x)x+xD(x)=0 for all x ∈U. Thus, we completed our proof. By using the similar techniques used in Theorem 2.1 one can prove the following theorem:

### *Theorem 3.8*

Let R be a 2-torsion free and 3-torsion free semiprime ring, U a nonzero left ideal of R such that rR(U)=0,and α:R→ R is an automorphism. If there exists an (α,α)-derivation D: R→ R such that [D(x),α(x)],α(x)] ∈Z(R), then D(R) ∈Z(R) and the ideal generated by D(R) is in the Z(R).

### Conclusions

It is clear from this study that, if D: N×N →Nis a symmetric mapping which also bi-additive (i.e., additive in both arguments), then the trace of D satisfies the relation d(x+y) = d(x)+2D(x,y)+d(y) for all x, y ∈N. A symmetric biadditive mapping D: N×N →N is called a symmetric biderivation if D(xy,z) = D(x,y)y+ xD(y,z) is fulfilled for all x, y, z ∈N. In this paper we investigate concerning a (α,α)- symmetric derivations D on semiprime rings and prime rings R.

### Conflicts of interest

Authors declare no conflict of interest.

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