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Research Article

Characterizations of Finite Semigroups of Multiple Operators

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Abstract

In the present paper, we studied Ω-monoids. We define and characterize the Ω-semigroups as a universal algebra which is a semigroup and in which there is given a system of binary operations Ω satisfying the associative condition: $((x, y), z)\beta = (x, (y, z)\beta)\alpha$ for all $x, y, z \in S$ and for each pair of binary operations α, β.

Keywords: Ω-semigroup; Finite derivation type; String-rewriting systems; Derivation graph; Homotopy.

Introduction

A monoid has finite derivation type (FDT) if the full homotopy relation is generated by a finite set called a homotopy base [1]. Squier proved that this property is indeed a property of finitely presented monoids, that is, it is an intrinsic property of a monoid independent of its presentation [2]. He established the fact that every monoid that can be presented through a finite convergent presentation does have FDT. Thus, FDT is one of the necessary conditions that a finitely presented monoid must satisfy in order that it can be presented by some finite convergent string-rewriting system. In this paper we generalize these results in the case of Ω-monoids [3].

We define, first, the Ω- semigroups as a universal algebra which is a semigroup and in which there is given a system of binary operations Ω satisfying the associative condition: $((x, y), z)\beta = (x, (y, z)\beta)\alpha$ for all $x, y, z \in S$ and for each pair of binary operations α, β [4]. In the first sections of the paper we define and give some general results related to the Ω-string rewriting systems, the properties of confluence, Noetherian, Church-Rosser, critical peaks, the word problem for the Ω-monoids and so on [5]. The last two sections are dedicated to the property of finite derivation type (FDT) and the related results of [6] generalized in the case of Ω- monoids.

Preliminaries

In this section we give some preliminaries which are useful in the sequel. We begin by the following definition.

Definition 2.1

A binary relation on X is a subset $R \subseteq X \times X$. If $(x, y) \in R$, then we denote this by xRy and we say that x is related to y by R. The inverse relation of R is the binary relation $R^{-1} \subseteq X \times X$ defined by $yR^{-1}x \Leftrightarrow (x, y) \in R$. The relation $IX = \{(x, x), x \in X\}$ is called the identity relation. The relation $(X)^2$ is called the complete relation [7, 8, 9].

Let $R \subseteq X \times X$ and $S \subseteq X \times X$ two binary relations. The composition of R and S is a binary relation $S \circ R \subseteq X \times X$ defined by $xS \circ Rz \Leftrightarrow \exists y \in X$ such that xRy and ySz.

A binary relation R on a set X is said to be

i. Reflexive if xRx for all $x \in X$;
ii. Symmetric if xRy implies yRx;
iii. Transitive if xRy and yRz imply xRz;

4. Antisymmetric if xRy and yRx imply $x = y$.

Let R be a relation on a set X. The reflexive closure of R is the smallest reflexive relation R^0 on X that contains R; that is,

i. $R \subseteq R^0$

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ii. If R' is a reflexive relation on X and $R \subseteq R'$, then $R^0 \subseteq R'$.

The symmetric closure of R is the smallest symmetric relation $R+$ on X that contains R; that is

i. $R \subseteq R+$
ii. If R' is a symmetric relation on X and $R \subseteq R'$ then $R+ \subseteq R'$.

The transitive closure of R is the smallest transitive relation $R*$ on X that contains R; that is

i. $R \subseteq R*$
ii. If R' is a transitive relation on X and $R \subseteq R'$ then $R* \subseteq R'$.

Let R be a relation on a set X. Then

i. $R^0 = R \cup IX$
ii. $R^+ = R \cup R^{-1}$
iii. $R^* = \bigcup Rk\ k{=}{+}\infty k{=}1$.

Let X be an alphabet. A semi-Thue system R over X, for briefly STS, is a finite set $R \subseteq X* \times X*$, whose elements are called rules [10]. A rule (s, t) will also be written as $s \rightarrow t$. The set (R) of all left-hand sides and $r(R)$ of all right-hand sides are defined as follows:

$(R) = \{s \in X*, \exists t \in X*: (s, t) \in R\}$ and $r(R) = \{t \in X*, \exists s \in X*: (s, t) \in R\}$.

If R is finite, then the size of R is denoted by $||R||$ and is defined as $||R|| = \Sigma\ (|s| + |t|)\ (s,)\in R$.

We define the binary relation $\rightarrow R$ as follows, where $u, v \in X*: u \rightarrow R\ v$ if there exist $x, y \in X*$ and $(r, s) \in R$ with $u = xry$ and $v = xsy$. We write $u \rightarrow R * v$ if there are words $u0, u_1, \dots, u_n \in X*$ such that $u0 = u, ui \rightarrow R\ ui{+}1, \forall 0 \leq i \leq n - 1, u_n = v$. If $n = 0$, we have $u = v$, and if $n = 1$, then we have $u \rightarrow R\ v$. Note that $\rightarrow R *$ is the reflexive transitive closure of $\rightarrow$. The Thue congruence $\leftrightarrow R *$ is the equivalence relation generated by $\rightarrow$. If R is a relation on $X*$ and $R\#$ denotes the congruence generated by R then the relations $\leftrightarrow R *$ and $R\#$ coincide. A decision problem is a restricted type of an algorithmic problem where for each input there are only two possible outputs. In other words, a decision problem is a function that associates with each input instance of the problem a truth value true or false.

Definition 2.2.

A graph G is a 5-tuple $G = (V, E, \sigma, \tau,\text{-}1)$, where V is the set of vertices and E is the set of edges of G; $\sigma, \tau: E \rightarrow V$ are mappings, which associate with each edge $e \in E$ its initial vertex $\sigma(e)$ and its terminal vertex $\tau(e)$, respectively.;and e^{-1}: $E \rightarrow E$ is a mapping satisfying the following conditions: $e^{-1} \neq e$, $(e^{-1})^{-1} = e$, $\sigma(e^{-1}) = \tau(e)$ and $\tau(e^{-1}) = \sigma(e)$ for all $e \in E$.

Definition 2.3

Let $G = (V, E, \sigma, \tau,\text{-}1)$ be a graph, and let $n \in \mathbb{N}$. A path in G (of length n) is a $(2n + 1)$-tuple $p = (v0, e_1, v_1, \dots, v_n^{-1}, e_n, v_n)$ with $v_0, v_1, \dots, v_n \in V$ and $e_1, e_2, \dots, e_n \in E$ such that $\sigma(e_i) = v_i{-}1$ and $\tau(e_i) = v_i$ hold for all $i = 1,2, \dots, n$. In this situation p is a path from v_0 to v_n, and the mappings σ, τ can be extended to paths by setting $(p) = v_0$ and $(p) = v_n$. For $u, v \in V$, (u, v) denotes the set of paths in G from u to v. In particular, for each $v \in V$, (v, v) contains the empty path (v).

Also the mapping -1 can be extended to paths. The inverse path $p{-}1 \in (vn, v0)$ of p is the following path $p^{-1} = (v_n, e_n^{-1}, vn{-}1, \dots, v1, e_1^{-1}, v0)$. Finally, if $p \in (u, v)$ and $q \in (v, w)$, then the composite path $p \circ q \in (u, w)$ is defined in the obvious way.

It is clear that, the composition of paths is an associative operation, and the empty paths act as identities for composition. Next, if $p \in (u, v)$, then $(p{-}1){-}1 = p$, and if $q \in P(v, w)$ then $(p \circ q){-}1 = q{-}1 \circ p{-}1$. Finally, if p is an empty path, then $p{-}1 = p$. If G is a graph, then $P(G)$ will denote the set of all paths in G, and $P(2)(G) = \{(p, q)|p, q \in P(G) such\ that\ \sigma(p) = \sigma(q) and\ \tau(p) = \tau(q)\}$ is the set of all pairs of paths that have a common initial vertex and a common terminal vertex.

Definition 2.4.

Let $G1 = (V1, E1, \sigma1, \tau1,\text{-}1)$ and $G2 = (V2, E2, \sigma2, \tau2,\text{-}1)$ be graphs. A mapping from $G1$ to $G2$ is an ordered pair $f = (fV, \ \ )$ of functions, where $fV : V1 \rightarrow V2$ and for each $e \in E1$, $fE\ (e)$ is a path in $G2$ from $fV\ (\sigma1(e))$ to $fV\ (\tau1(e))$. Further, for each $e \in E1$, $fE\ (e{-}1) = (fE\ (e)){-}1$. The mapping f is called a morphism if fE carries edges to edges.

It is clear that a mapping f: $G1 \rightarrow G2$ induces a mapping f: $(G1) \rightarrow (G2)$.

Definition 2.5.

Let $G = (V, E, \sigma, \tau,\text{-}1)$ be a graph. A subgraph $G1 = (V1, E1, \sigma1, \tau1,\text{-}1)$ of G consists of a subset $V1$ of V and a subset $E1$ of E such

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that, for all $e \in E1$, $\sigma 1(e) = \sigma(e) \in V1$ and $\tau 1(e) = \tau(e) \in V1$. Next, $e{-}1 \in E1$ for all $e \in E1$.

Definition 2.6.

([6]) A type of universal algebras is an ordered pair of a set T and a mapping $\omega \mapsto n\omega$ that assigns to each $\omega \in T$ a nonnegative integer $n\omega$, the formal arity of ω. A universal algebra, or just algebra of type T is an ordered pair of a set A and a mapping, the type – T algebra structure on, that assigns to each $\omega \in T$ an operation ωA on A of arity $n\omega$.

Results and discussion

A semigroup with multiple operators or a Ω-semigroup is a universal algebra which is a semigroup and in which there is given a system of binary operations Ω satisfying the associative condition: $((x, y), z) = (x, (y, z))$ for all $x, y, z \in S$ and for each pair of binary operations α, β. Let (S, Ω), (T, Ω) be two Ω-semigroups. Then, f: $S \rightarrow T$ is a homomorphism if $((x, y)) = ((x), (y))$, $x, y \in S$, $\forall \omega \in \Omega$. Next, we define the free Ω-semigroup using the concept of the free word algebra of a type T with the set X as basis, as it is described in [6]. For the case of Ω-semigroups, we agree, first, that their type is simply a set of binary relations which we denote by Ω. So, we construct, inductively, the free Ω-word algebras as follows: denote $W0 = X$, then for $k > 0$ denote Wk the set of all sequences $(\gamma, w1, w2)$ where $w1, w2 \in Wk{-}1$ and $\gamma \in \Omega$. For each $\alpha \in \Omega$, we denote by $\lambda\alpha$ the empty word related to α. Now, we take $WX = \cup Wkk{\geq}0$. Writing this in letters, we will have that $W1$ is the set of all sequences (γ, x, y) where $\gamma \in \Omega$ and $x, y \in X$. It is more convenient to denote these sequences in the form $x\gamma y$. The product $x\beta\lambda\beta$ is defined to be x, and similarly the product of the form $\lambda\alpha\alpha y$ is defined to be y, where, $\lambda\beta$ are the empty words related to the operators α, β, respectively. In the next step, $W2$ would have as elements the sequences $(\gamma, w1, w2)$ where $w1, w2 \in W1$ and $\gamma \in \Omega$. If $w1 = x1\gamma 1y1$ and $w2 = x2\gamma 2y2$, then $(\gamma, w1, w2)$ would be just the sequence $x1\gamma 1y1\gamma x2\gamma 2y2$, with our new notations. And this procedure continues.

Example 3.1

A semigroup is a set with a single binary operation. Here Ω consists of a single element μ of arity two such that the following associative law is satisfied $xy\mu z\mu = xyz\mu\mu$ for all $x, y, z \in S$.

Example 3.2

A Γ-semigroup is a special case of an Ω-semigroup. Indeed, we define in S binary operators $\bar{\alpha}$: $S \times S \rightarrow S$ such that $\bar{\alpha}(x, y) = x\alpha y$, $\forall \alpha \in \Gamma$. Then, $(S, \bar{\Gamma})$ is a Ω-algebra where $\bar{\Gamma} = \{\bar{\gamma}: \gamma \in \Gamma\}$ satisfying the conditions $\bar{\beta}(\bar{\alpha}(x, y), z) = \bar{\alpha}(x, \bar{\beta}(y, z))$, $\forall x, y, z \in S$, $\bar{\alpha}, \bar{\beta} \in \bar{\Gamma}$.

Example 3.3

It is clear that the free Ω-semigroup defined as above is a Ω-semigroup. We will denote with $MX{*}\Omega$ the free Ω-monoid on X, that is the set of finite products $x1\gamma 1 \ldots xn{-}1\gamma n{-}1xn$ with $x1, \ldots, xn \in X$, $\gamma i \in \Omega$, i = 1,2, … , n − 1, including the empty product 1.
It is the smallest Ω-submonoid of M containing X.

If $MX{*}\Omega = M$, we say that X generates M, or that X is a set of generators for M. If X is finite and generates M, we say that M is a finitely generated Ω-monoid. If X generates M and no strict subset of X does, we say that X is a minimal set of generators for M.

Theorem 3.4

If M is a finitely generated Ω-monoid and X is a set of generators for M, then there is a finite subset of X which generates M. In particular, any minimal set of generators for M is finite.

Proof:

Indeed, for any $y = x1\gamma 1 \ldots xn{-}1\gamma n{-}1xn \in M$ with $x1, \ldots, xn \in X$, $\gamma \in \Omega$, we get a finite set $X(y) = \{x1, \ldots, xn\} \subset X$. If $Y = \{y1, \ldots, ym\}$ generates M, so does the finite set $X(Y) = X(y1) \cup \ldots \cup X(ym) \subset X$. Now, if M is a Ω-monoid, then any map f: $X \rightarrow M$ extends to a unique morphism $\bar{f}$: $MX{*}\Omega \rightarrow M$. A presentation is a pair $(X; R)$ where X is an alphabet and R is the following set $R = \{(u, v)| u, v \in \}$. The congruence generated by R is defined as follows:
i. $u\alpha u'\beta v \leftrightarrow R\ u\alpha v'\beta v$ whenever $u, v \in MX{*}\Omega$, $\alpha, \beta \in \Omega$, and $u'Rv'$
ii. $x \leftrightarrow R * y$ whenever $x = x0 \leftrightarrow R\ x1 \leftrightarrow R \ldots \leftrightarrow R\ xn = y$.
We denote by MR the quotient $MR = MX{*}\Omega/{\leftrightarrow}R*$ which is a Ω-semigroup.

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Indeed, it easily verified that the congruence generated by R, as we defined it, is a Ω-congruence. For this, it's enough to see that $u\alpha u'\beta v \leftrightarrow R\ u\alpha v'\beta v \Rightarrow u\alpha u'\beta v\gamma w \leftrightarrow R\ u\alpha v'\beta v\gamma w$ and $u\alpha u'\beta v \leftrightarrow R\ u\alpha v'\beta v \Rightarrow w\gamma u\alpha u'\beta v \leftrightarrow R\ w\gamma u\alpha v'$. Let us denote shortly by ρ this congruence. Now, for $u\rho, v\rho \in MR$ and $\gamma \in \Omega$, let $(u\rho)(v\rho) = (u\gamma v)\rho$. This is well-defined, since for all $u, v \in MX{*}\Omega$ and $\gamma \in \Omega$, $u\rho = u'\rho$ and $v\rho = v'\rho \Rightarrow (u, u'), (v, v') \in \rho \Rightarrow (u\gamma v, u'\gamma v), (u'\gamma v, u'\gamma v') \in \rho \Rightarrow (u\gamma v, u'\gamma v') \in \rho \Rightarrow (u\gamma v) = (u'\gamma v')\rho$. Let $u, v, w \in MX{*}\Omega$ and $\gamma, \mu \in \Omega$. Then, it follows that $(u\rho\gamma v\rho)\mu w\rho = ((u\gamma v)\rho)\mu w\rho = ((u\gamma v)\mu w)\rho = (u\gamma(v\mu w))\rho = u\rho\gamma(v\mu w)\rho = u\rho\gamma(v\rho\mu w\rho)$ and this result completes the proof.

We have a canonical surjection : $MX{*}\Omega \rightarrow MX{*}\Omega/\leftrightarrow R\,*$ as well. Moreover, if $f\colon X \rightarrow M$ is a map such that $(x) = (y)$ whenever xRy and $\bar{f}: MX{*}\Omega \rightarrow M$ its extension we obtain a unique morphism $\tilde{f}: MX{*}\Omega/\leftrightarrow R\,* \rightarrow M$ such that $\tilde{f} \circ \pi R = \bar{f}$. If the map $\tilde{f}$ is bijective, we write $M \cong MX{*}\Omega/\leftrightarrow R\,*$ and we say that $(X; R)$ is a presentation of the Ω-monoid M. This means that the set (X) generates M, and that $\bar{f}(x) = \bar{f}(y)$ if and only if $x \leftrightarrow R * y$. If the map $\tilde{f}$ is bijective and both X and R are finite we say that M is a finitely presented Ω-monoid. And again, if the map $\tilde{f}$ is bijective, (X) is a minimal set of generators and no strict subset of R generates the congruence $\leftrightarrow R*$, then we say that $(X; R)$ is a minimal presentation of M.

Corollary 3.5

For any morphism: $MX{*}\Omega/\leftrightarrow R\,* \rightarrow MY{*}\Omega/\leftrightarrow S*$, there is a morphism φ: $MX{*}\Omega \rightarrow MY{*}\Omega$ such that $\pi S \circ \varphi = f \circ \pi R$.

Proof: $MX{*}\Omega\ \varphi\!\rightarrow\ MY{*}\Omega$, $\pi R \downarrow \downarrow \pi S$ and $MX{*}\Omega/\leftrightarrow R * f \rightarrow MY{*}\Omega/\leftrightarrow S*$. It is sufficient to define (x) for each $x \in X$, and for this we have to use the fact that πS is surjective.

As a crucial step, we define the derivations for the presentation as follows:

i) An atomic derivation $r\ A\!\rightarrow s$ is given by a pair $(r, s) \in R$,

ii) An elementary derivation $x\ E\!\rightarrow y$ is given by two words $u, v \in MX{*}\Omega$ and an atomic derivation $r\ A\!\rightarrow s$ such that $x = u\alpha r\beta v$ and $y = u\alpha s\beta v$. If $u = v = 1$, we identify E with the atomic derivation A,

iii) A derivation $x\ F\!\rightarrow y$ is given by a sequence $x = x0\ E1\!\rightarrow x1\ E2\!\rightarrow \ldots\ En \rightarrow xn = y$ of elementary derivations. If $n = 1$, we identify F with the elementary derivation $E1$. If $n = 0$, we get the identity derivation.

Composition of derivations is defined in obvious way. Also, if x, y are words and $z\ F\!\rightarrow z'$ is a derivation, the derivation $x\alpha z\beta y\ xFy \rightarrow x\alpha z'\beta y$ is defined in the obvious way.

Let $(X; R)$ be a Ω-monoid presentation such that the Ω-string-rewriting system R is noetherian. This means that there is no infinite sequence $x0\ E1\!\rightarrow x1\ E2\!\rightarrow \ldots\ En \rightarrow xn\ En{+}1 \rightarrow \ldots$ of elementary derivations. Then for any $\in MX{*}\Omega$, there is a derivation $x\ F\!\rightarrow y$ where y is reduced which means that no elementary derivation starts from y. This y is called a normal form of x.

A peak is an unordered pair of elementary derivations $x\ E\!\rightarrow y$ and $x\ E' \rightarrow y'$ starting from the same word x. Such a peak is called confluent if there is a word z and two derivations $y\ F\!\rightarrow z$ and $y'\ F' \rightarrow z$. It is called critical if $E \neq E'$ and if it is of the form $r\alpha v = u'\alpha' r'$ where, in the first case, u' is a strict prefix of r, or equivalently, v is a strict suffix of r'.

Theorem 3.6

If $(X; R)$ is a finite convergent presentation then $\leftrightarrow R\,*$ is a decidable relation.

Proof:

It would be enough to compare the reduced form which, in this case, are obviously computable. If $\leftrightarrow R\,*$ is a decidable relation then we say that that the Ω-monoid M has a decidable word problem and this property does not depend on the choice of the presentation as long as this presentation is finitely generated, i.e. X is finite. Indeed, assume that $(X; R)$ and $(Y; S)$ are finitely generated presentations of the Ω- monoid M such that $MR \cong M \cong MS$. Then for every $a \in X$ there exists a word $wa \in MY{*}\Omega$ such that a and wa represent the same element of M. If we define the homomorphism h: $MX{*}\Omega \rightarrow MY{*}\Omega$ by $h(a) = wa$ then for all $u, v \in MX{*}\Omega$ we have $u \leftrightarrow R * v$ if and only if $h(u) \leftrightarrow R * h(v)$. Thus the word problem for $(X; R)$ can be reduced to the word problem for $(Y; S)$ and vice versa. Thus the decidability and complexity of the word problem does not depend on the chosen presentation. Hence, we may just speak of the word problem for the Ω-monoid M.

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Theorem 3.7

Convergence is a decidable property for any finite noetherian presentation.

Proof:

It follows from the facts that there are finitely many critical peaks in this case and is easily seen that they are computable.

Conclusions

In the present paper we have shown that if $(X; R)$ is a presentation of a Ω-monoid, each $\rho = (x, y) \in R$ can be seen as a rewrite rule $x\ \rho\!\to y$, with source x and target y. An elementary reduction is of the form $u\alpha x\beta v\ u\rho v \to u\alpha y\beta v$ where u, v are words and $x\ \rho\!\to y$ is a rule (as we define it) . A reduction is a finite sequence $x = x0\ r1\!\to x1\ r2\!\to x2\ \ldots\ xn{-}1\ rn\ \to xn = y$ of elementary reductions. Each rule is considered as an elementary reduction, and any elementary reduction is considered as a reduction of length 1. If $x\ r\!\to y$ and $y\ s\!\to z$ are reductions, we write $r * s$ for the composed reduction $x\ r\!\to y\ s\!\to z$. Furthermore, there is an empty reduction $\to x$ for any word $x \in MX{*}\Omega$. So we obtain a category of reductions $(X;\ R)$. We call R a Ω-string rewriting system.

Conflicts of interest

Authors declare no conflict of interest.

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