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Research Article

On Normal Intersection Conjugacy Functions in Finite Groups

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Abstract

In the present paper, we investigate conjugacy classes of subgroups of a fixed but arbitrary group where the definition of these subgroups is made entirely within the structure of the group. We have shown that these subgroups are analogous to F-injectors of the group where F is a locally defined Fitting class, but preclude the existence of such a class in their definition. Moreover, the subgroups used to define the subgroups of such a conjugacy class are analogous to the F(p)-radicals of the group.

Keywords: Normal; Intersection; Conjugacy; Functions; Finite Group; Solvability.

Introduction

Recent work of [1-3] have suggested an internal approach to certain problems that have involved global considerations. It is the purpose of this paper to continue the investigation of these problems in the same manner. All groups considered are assumed to be finite. Fischer introduced the notion of a Fitting class of groups [4]; that is, an isomorphism closed class F of groups such that if $G \in F$ and N G, then $N \in F$, and if $N_i \in F$ and $N_i \leq G$ for $i = 1, 2,\ldots$, then $N_1N_2 \in F$. From this definition it is clear that every group G has, for every such F, a unique normal subgroup that is maximal with respect to belonging to F [5]. This subgroup is called the F-radical and is denoted by G_F. If F is any Fitting class, then an F-injector of a group G is a subgroup V of G such that $V \cap N$ is a maximal F-subgroup of N {i.e., $V \cap N \in F$ and $V \cap N = L$ whenever $V \cap N \leq L \leq N$ and $L \in F$) for every subnormal subgroup N of G.

In [6] it was shown that every solvable group possesses a unique conjugacy class of F-injectors for every Fitting class F of solvable groups. The work in [7] introduces the notion of a locally defined Fitting class F; that is, F equals to where, for each prime p, F(p) is a Fitting class. This, of course, may easily be considered to be a kind of dual of a locally defined formation X; that is , where, for each prime p, X(p) is a formation. A formation is an isomorphism closed class $\mathcal{Y}$ of groups such that if $G \in \mathcal{Y}$ and N G, the $G/N \in \mathcal{Y}$, and if $G/N_i \in \mathcal{Y}$ for i = 1,2, then $G/N_1 \cap N_2 \in \mathcal{Y}$. On a more basic level, formations and Fitting classes may be considered duals of each other [8].

The normal subgroup closure property of Fitting classes may be thought of as a dual notion of the homormorphic image closure property of formations. Likewise, the normal product closure property of Fitting classes may be thought of as a dual notion of the subdirect product closure property of formations. The properties of a Fitting class motivate the definition of a type of function, for each prime p, from the set of subgroups of a fixed but arbitrary group G to that set again. The set of the subgroups in the images of the functions satisfies properties similar to those of a Fitting class. These subgroups are then used to define a set of subgroups of G that is analogous to the set of subgroups of G contained in a locally defined Fitting class. It is then shown that if, for certain normal subgroups in the images of the functions, G modulo each of those normal subgroups meets certain requirements generally less restrictive

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than the solvability of G, then G possesses a unique conjugacy class of subgroups, analogous to a conjugacy class of F-injectors for some Fitting class F, and possessing similar properties. The requirements which the factor groups mentioned above must possess are closely related to one used in [9] in extending the class of groups which have a unique conjugacy class of nilpotent injectors beyond solvable groups.

One of the results of this paper is that the injector-like subgroups arising from the functions discussed earlier are completely determined by the nilpotent injectors of a certain factor group of G. It is also demonstrated that while every locally defined Fitting class may be interpreted as a set of appropriate functions, the converse does not hold. We also extend the internal approach to the results on homomorphs. A homomorph is an isomorphism closed class $\mathcal{H}$ of groups such that every homomorphic image of a group in $\mathcal{H}$ lies again in $\mathcal{H}$. A homomorph $\mathcal{H}$ is said to be primitively closed if $G \in \mathcal{H}$ whenever $G/\mathrm{core}_G(M) \in \mathcal{H}$ for every maximal subgroup M of G. A subgroup S of G is said to be an $\mathcal{H}$-projector of G is $S \in \mathcal{H}$ and $L = SL_0$ whenever $S \leq L \leq G$, L_0 L and $L/L_0 \in \mathcal{H}$. (The definition is the same when $\mathcal{H}$ is a formation).

The author in [10] defines a homomorph of solvable groups to be saturated if every solvable group has an $\mathcal{H}$-projector. A result of [3] shows that a homomorph $\mathcal{H}$ of a solvable group is saturated if and only if it is primitively closed. The relationship between $\mathcal{H}$-projectors and primitive closure is exploited in the definition of a type of function that ultimately yields the desired results. Given a fixed but arbitrary group G, this type of function maps ordered pairs, each consisting of a subgroup of G and a conjugacy class of maximal subgroups of that subgroup, to subgroups of G. The set of image subgroups possesses properties similar to a homomorph. Again the image subgroups are used to define a set of subquotients of G analogous to the set of subquotients of G lying in some primitively closed homomorph. If the group G satisfies certain conditions, dependent upon the function and generally less restrictive than solvability, then G is shown to possess a unique conjugacy class of subgroups with properties similar to those of an $\mathcal{H}$-projector for some primitively closed homomorph F. As a duel of an F-radical, one sees from the definition of a formation X that for every group G there exists a unique normal subgroup N of G such that G/N is the largest factor group of G lying in X. That subgroup is called the X-residual of G and is denoted by G^X. For a locally defined formation.

In [2] they investigated a generalization of system normalizers, namely X-normalizers. An X-normalizer of a solvable group G is a subgroup D of G of the form. These subgroups were shown to have properties similar to those of system normalizers: they form a unique conjugacy class; for each prime p, they cover the p-chief factors of G centralized by and avoid the p-chief factors of G not centralized by; and they are characterized as minimal members of certain chains of subgroups of G. It is shown in [5] that there exists a solvable group G and primitively closed homomorph $\mathcal{H}$ such that $G \notin \mathcal{H}$ but G has no subgroups of the type that characterize formation normalizers. Thus, [2] defines homomorph normalizers as minimal members of certain chains of subgroups, similar to the characterization of formation normalizers; and then restricts the class of groups under consideration to those satisfying the appropriate condition regarding the existence of the necessary subgroups. For a fixed but arbitrary solvable group G, reserchers have defined separate types of functions from the set of subgroups of G into the same set. In each case they defined a set of subquotients of G analogous to the set of subquotients of G contained in some locally defined formation. They then proved the existence of subgroups in the respective sets of subquotients of G with properties similar to those of either formation projectors or normalizers.

Next, we show that each such function induces a function of the type considered in this work in a natural way. We note that the normalizers considered are defined as minimal members of certain chains of subgroups. In addition to this method of definition of formation normalizers and the one mentioned earlier, the author in [7] defined formation normalizers as the formation projectors of certain subgroups of G. In the solvable case, these three methods of definition yield the same conjugacy class of subgroups. The underlying emphasis of this paper is a return to an internal approach to questions heretofore given global consideration.

In classical formation theory or Fitting class theory, the typical theorem schema is: "If F is a formation of solvable groups, then every solvable group…", or "If F is a Fitting class of solvable groups, then every solvable group…". By eliminating the large class of groups, we are able to consider the classical questions on a group-by-group basis with the theorems bearing the typical schema: "If G is a group, X a certain set of functions from the set of subgroups of G into the same set, and G satisfies certain conditions dependent only upon X, then…".

Research methodology

In this section we give techniques used to investigate conjugacy classes of subgroups of a fixed but arbitrary group where the definition of these subgroups is made entirely within the structure of the group. These subgroups are analogous to F-injectors of the group where F is a locally defined Fitting class, but preclude the existence of such a class in their definition. The subgroups used to define the subgroups of such a conjugacy class are analogous to the F(p)-radicals of the group, but again obviate the need for such Fitting classes. These defining subgroups are then used to characterize the injector-like subgroups as subgroups that are maximal with respect to a certain property. In the case of F-injectors of a solvable group, where F is a locally defined Fitting class, this property is "maximal with respect to being an F-subgroup containing the F-radical." This property is shown to be equivalent to being a maximal nilpotent subgroup of a certain factor group containing the Fitting subgroup of that factor group. It is shown in [4] that this property characterizes nilpotent injectors; thus these subgroups, modulo a certain normal subgroup of the group, are precisely the nilpotent injectors of that factor group.

The defining subgroups are also used to find conditions on the group, usually less restrictive than solvability, for which the usual results remain valid. The conditions are closely related to nilpotent constraint (i.e., $C_G(F(G)) \leq F(G)$) which was used by [8] in the characterization of nilpotent injectors mentioned above. The subgroups used in the definition of the subgroups of the conjugacy classes will be the images of certain functions which in a sense select the F(p)-radicals of the subgroups. Indeed, any Fitting class F induces this type of function by mapping each subgroup to its F-radical. However, example 2.4.b demonstrates the existence of such functions which do not arise in this way from any Fitting class. Thus the type of functions defined below may be thought of as a generalization of the radical in the classical theory.

Definition 2.1.

Let G be a group. A normal intersection conjugacy functions of G (NICF of G) is a function Y from $\{H \mid H \leq G\}$ into itself satisfying the following conditions:

i. for every $H \leq G$, $Y(H)$ H;
ii. for every $H \leq G$ and $g \in G$, ;
iii. if K H $\leq$ G, then $Y(K) = Y(H) \cap K$.

(We note that condition ii and $Y(H) \leq H$ imply $Y(H)$ H.)

Definition 2.2.

Let G be a group. A normal intersection conjugacy function system of G is a collection $\{Y_p\}$ of NICF's of G, one for each prime p.

Definition 2.3.

Let $\mathcal{Y} = \{Y_p\}$ be an NICF system of a group G. Then define $\mathcal{Y}(G) = \{Y_p(H) \mid Y_p \in \mathcal{Y}$ and $H \leq G\}$ and call $\mathcal{Y}(G)$ the image of $\mathcal{Y}$ in G.

Example 2.4.

Given a group G, we define an NICF, Y_p, of G for each prime p by $Y_p(H) = O_{p'}$, p(H) for each $H \leq G$. Then is clearly an NICF system of G.

b) Let K be a noncyclic group of order 4. Let and define Y by and. Then Y is an NICF of K, but for every $H \leq K$, Y(H) is not the F-radical where F is a Fitting class since $G \in F$ whenever .

Definition 2.5.

Let $\mathcal{Y}$ be an NICF system of a group G. Then define . Call the $\mathcal{Y}$-set of G.

Remark 2.6.

If is a locally-defined Fitting class of solvable groups, then for a solvable group G, $\{H \mid H \leq G$ and $H \in F\} = \{H \mid H \leq G$ and for each p, $H/H_{F(p)} \in \mathcal{S}_{p'} \mathcal{S}_p\} = \{H \mid H \leq G$ and for each p, $H/H_{F(p)}$ is p-nilpotent$\}$. Thus is meant to correspond to the set of subgroups of G contained in a locally defined Fitting class.

Every locally defined Fitting class gives rise in a natural way to an NICF system of a group G. For each prime p, define a function Y_p by where H is a subgroup of G and is the $\mathcal{H}$(p)-radical of H. Then it is easy to see that $\mathcal{Y} = \{Y_p\}$ is an NICF system of G and for a subgroup H of G, H $\in \mathcal{H}$ if and only if H $\in$.

However, there is a group G and NICF system $\mathcal{Y}$ of G such that there is no Fitting class F such that for H $\le$ G, if and only if H $\in$ F. The following example also shows that may not be isomorphism closed.

Example 2.7.

Let, the direct product of two copies of the symmetric group on three elements. We define an NICF system of G in the following way:
$\mathcal{Y} = \{Y_p \mid Y_p(G) = G$ for $p \neq 3$ and $Y_3(G) = \Sigma_3$, and for $H \le G$, $Y_p(H) = Y_p(G) \cap H\}$. Then and since * is not 3-nilpotent. However, $Y_3(\Sigma_3) = \Sigma_3$ and $Y_2(\Sigma_3) = \Sigma_3$, and so . Thus is not isomorphism closed, and hence there is no Fitting class F such that $\{H \mid H \le G \text{ and } H \in \mathbf{F}\} = \bar{\mathcal{Y}}(G)$.

Definition 2.8.

Let $\mathcal{Y}$ be an NICF system of a group G. For each subgroup H of G define $H_{\mathcal{Y}} = \prod\{N \mid N \trianglelefteq H \text{ and } N \in \bar{\mathcal{Y}}(G)\}$ and call $H_{\mathcal{Y}}$ the $\mathcal{Y}$-radical of H. (We note that $H_{\mathcal{Y}} \in \bar{\mathcal{Y}}(G)$ by Theorem 3.1. below).

Results and Discussion

We now show that for a give NICF system $\mathcal{Y}$ of G, has certain properties in common with Fitting classes.

Theorem 3.1.

Let $\mathcal{Y}$ be an NICF system of G. Then the following hold.

1) If $N \trianglelefteq H \le G$ and $H \in \bar{\mathcal{Y}}(G)$, then $N \in \bar{\mathcal{Y}}(G)$.

2) If $N_i \trianglelefteq H \le G$ and $N_i \in \bar{\mathcal{Y}}(G)$ for i = 1, 2, then $N_1N_2 \in \bar{\mathcal{Y}}(G)$.

Proof. 1:

If $H \in \bar{\mathcal{Y}}(G)$, then for each prime p and $Y_p \in \mathcal{Y}$, $H/Y_p(H)$ is p-nilpotent. Since $\frac{N}{Y_p(N)} = \frac{N}{Y_p(H) \cap N} \cong \frac{NY_p(H)}{Y_p(H)}$, we have for each p, $N/Y_p(N)$ is p-nilpotent and so $N \in \bar{\mathcal{Y}}(G)$.

2). If $N_i \in \bar{\mathcal{Y}}(G)$ for i = 1, 2, then for each p and $Y_p \in \mathcal{Y}$, $\frac{N_i}{Y_p(N_i)}$ is p-nilpotent.

Since $N_i \trianglelefteq N_1N_2$, $Y_p(N_1N_2) \cap N_i = Y_p(N_i)$, $i = 1,2$.

Thus $\frac{N_i}{Y_p(N_1N_2) \cap N_i} \cong \frac{N_iY_p(N_1N_2)}{Y_p(N_1N_2)}$ is p-nilpotent and so $\frac{N_1N_2}{Y_p(N_1N_2)} = \prod_{i=1}^{2} \frac{N_iY_p(N_1N_2)}{Y_p(N_1N_2)}$ is p-nilpotent.

Thus $N_1N_2 \in \bar{\mathcal{Y}}(G)$. This completes the proof.

Q.E.D.

Theorem 3.2.

Let $\mathcal{Y}$ be an NICF system of G.

1) If $H \trianglelefteq G$, then for each $Y_p \in \mathcal{Y}$, $Y_p(H) \trianglelefteq G$.

2) For every subgroup H of G and $Y_p \in \mathcal{Y}$, $Y_q \in \mathcal{Y}$, we have $Y_q(Y_p(H)) = Y_p(Y_q(H)) = Y_p(H) \cap Y_q(H)$. In particular, $Y_p(Y_p(H)) = Y_p(H)$.

3) For every subgroup H of G and g $\in$ G, $(H_{\mathcal{Y}})^g = (H^g)_{\mathcal{Y}}$.

4) For $K \trianglelefteq H \le G$, $K_{\mathcal{Y}} = H_{\mathcal{Y}} \cap K$.

Proof. 2:

For every g$\in$G and $Y_p \in \mathcal{Y}$, $Y_p(H) = Y_p(H^g) = [Y_p(H)]^g$.

2) Since $Y_p(H) \trianglelefteq H$, we have $Y_q(Y_p(H)) = Y_q(H) \cap Y_p(H)$.

Similarly,

$Y_q(H) \trianglelefteq H$ implies $Y_p(Y_q(H)) = Y_p(H) \cap Y_q(H)$.

3) Since $H_{\mathcal{Y}}/Y_p(H_{\mathcal{Y}})$ is p-nilpotent, $\dfrac{(H_{\mathcal{Y}})^g}{[Y_p(H_{\mathcal{Y}})]^g}$ is p-nilpotent and so $\dfrac{(H_{\mathcal{Y}})^g}{Y_p[(H_{\mathcal{Y}})^g]}$ is p-nilpotent. Thus $(H_{\mathcal{Y}})^g \leq H^g$ and $(H_{\mathcal{Y}})^g \in \bar{\mathcal{Y}}(G)$ and so $(H_{\mathcal{Y}})^g \leq (H^g)_{\mathcal{Y}}$. By the same argument we have $\left[(H^g)_{\mathcal{Y}}\right]^{g^{-1}} \leq \left[(H^g)^{g^{-1}}\right]_{\mathcal{Y}} = H_{\mathcal{Y}}$. Thus $(H^g)_{\mathcal{Y}} \leq (H_{\mathcal{Y}})^g$ and therefore $(H^g)_{\mathcal{Y}} = (H_{\mathcal{Y}})^g$ as desired.

4) Since $H_{\mathcal{Y}} \cap K \leq H_{\mathcal{Y}}$, $H_{\mathcal{Y}} \cap K \in \bar{\mathcal{Y}}(G)$ and so $H_{\mathcal{Y}} \cap K \leq K_{\mathcal{Y}}$. By 3) above, for every h ∈ H, $(K_{\mathcal{Y}})^h = (K^h)_{\mathcal{Y}} = K_{\mathcal{Y}}$. Thus K is normal in H and so $K_{\mathcal{Y}} \leq H_{\mathcal{Y}} \cap K$. Therefore $K_{\mathcal{Y}} = H_{\mathcal{Y}} \cap K$. This completes the proof.

Conclusions

The present paper characterizes the notion of a locally defined Fitting class of solvable groups; that is, a Fitting class of the form where F(p) is a Fitting class of solvable groups for each prime p. As mentioned in the introduction, this seems a natural way to dualize the concept of a locally defined formation of solvable groups; that is, a formation which may always be written in the form where $\mathcal{H}$(p) is a formation of solvable groups for each prime p. In this chapter we investigate conjugacy classes of subgroups of a fixed but arbitrary group where the definition of these subgroups is made entirely within the structure of the group. These subgroups are analogous to F-injectors of the group where F is a locally defined Fitting class, but preclude the existence of such a class in their definition. The subgroups used to define the subgroups of such a conjugacy class are analogous to the F(p)-radicals of the group, but again obviate the need for such Fitting classes. These defining subgroups are then used to characterize the injector-like subgroups as subgroups that are maximal with respect to a certain property.

Conflicts of interest

Authors declare no conflict of interest.

References

[1] Conway JH, Curtis RT, Norton SP, Parker RA, Wilson RA. Atlas of finite groups. Oxford Univ. Press (Clarendon), Oxford and New York 1985.

[2] Guo X, Li S, Flavell P. Finite groups whose abelian subgroups are TI-subgroup. J Algebra 2007;307:565-9.

[3] Huppert B. Endliche Gruppen I Grundl. d. math. Wiss. 134. Springer-Verlag, Berlian Heidelberg, New York; 1979.

[4] Huppert B, Blackburn N. Finite groups III. Springer-Verlag, Berlin Heidelberg, New York; 1982.

[5] Isaacs IM. Character Theory of Finite Groups. Academic Press, New York; 1976.

[6] Li S, Guo X. Finite p-groups whose abelian subgroups have a trivial intersection. Acta Math Sin 2007;23(4):731-4.

[7] Walls G. Trivial intersection groups. Arch Math 1979;32:1-4.

[8] Williams JS. Prime graph components of finite groups. J Algebra 1981;69:487-513.

[9] Vijayabalaji S, Sathiyaseelan N. Interval-Valued Product Fuzzy Soft Matrices and its Application in Decision Making, Int J Mod Sci Technol 2016;1(7):159-63.

[10] Chinnadurai V, Bharathivelan K. Cubic Ideals in Near Subtraction Semigroups. Int J Mod Sci Technol 2016;1(8):276-82.

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