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Research Article

On Characterization of Very Rotund Banach Spaces

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Abstract

It is known that the Hilbert space *H* is the most rotund space among all Banach spaces. The question whether if a normed space *X* is a rotund Banach space implies we can obtain other most rotund spaces is still open and represents one of the most interesting and studied problems. In this paper we investigate if there exists other most rotund Banach spaces. It is shown that Frechet spaces are very rotund and also uniformly rotund.

Keywords: Rotundity; Hilbert space; Modulus; Convexity; Frechet space.

Introduction

Rotundity in Banach spaces has been studied over a period of time. Various notions of rotundity have been considered with very interesting results obtained [1-3]. Let *X* be a Banach space and a Hilbert space is a special type of Banach spaces. The following questions have over 20 years of history [4]. The standard method of measuring" the rotundity" of the unit ball in *X* is through the modulus of convexity δ_X : (0,2) → (0,1) of *X* $\delta_X(\varepsilon) : inf\{1- \| \frac{x+y}{2} \| : \| x \| \leq 1, \| \leq \leq y \| \leq 1, \| x - y \| \geq \varepsilon\}$ and the characteristic of convexity $\varepsilon_0(x) = Sup\{\varepsilon : \delta_X(\varepsilon) = 0\}$. The modulus of convexity has"two- dimensional character, "meaning that $\delta_X(\varepsilon) = inf\ \{\delta_E(\varepsilon): E \subset X, dim\ E = 2\}$[5]. It is known that the Hilbert space *H* is the most rotund space among all Banach space *X*, in the sense that $\delta_X(\varepsilon) \leq \delta_H(\varepsilon) = \delta_{E_2}(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$, where E_2 is the two-dimensional Euclidean space. Now, fix $a \in (0,2)$ and consider the class ε_a of all two-dimensional spaces $(E, \| . \|)$ having $\varepsilon_0(E) = a$. Which of these spaces is the most rotund? It can be formulated in the following questions: Given $\varepsilon \in (a,2)$, what is $Sup\{\delta_E(\varepsilon) : E \in \varepsilon_a\}$? Is there a space $E_a \in \varepsilon_a$: $\delta_E(\varepsilon) \leq \delta_{Ea}(\varepsilon) \forall E \in \varepsilon_a$? Moreover if the answer to these questions are positive, is such a space E_a in some sense unique?. The author in [6] considered spaces L^P and l^P for $P > 1$ and proved that they are uniformly rotund. The author in [7] later considered the Banach products of l^P type.

The study of [8] expanded the Days results, enlarged on the family of uniformly rotund Banach spaces and established analogous results for the notion of rotundity of the norm in a Banach space. In [9] the author considered a countable family of Banach spaces, $\{X_n : n \in \mathbb{N}\}$. In [10] they proved three characteristics of the very rotund space and discussed the relationship between the very rotund space and the geometrical properties of Banach space. In our case, the aim is to characterize uniform and most rotundity in Frechet spaces. Let us consider the following definitions.

Research Methodology

Definition 2.1

[4, Definition 1.1] A Banach space $(X, \|.\|)$ is Rotund(R) if given $x,y \in S_X$ with $x \neq y$, then $\| \frac{x+y}{2} \| < 1$.

Definition 2.2

[7, Definition 1.2] Let $(X, \| . \|)$ be a Banach space. $(X, \| . \|)$ is Uniformly Rotund (UR) if, given $\varepsilon > 0$, there exists $\delta > 0$ such that

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$\| \frac{x+y}{2} \| < 1-\delta$, whenever ‖δ : (0,2) → (0,1), defined by δ(ε) = 2 x-y , is called the Modulus of Rotundity(convexity) of the space X.

Definition 2.3

[2, Definition 1.3] Let $\{X_n : n \in N\}$ be a family of Banach spaces with the property (UR) and let δ_n be the modulus of rotundity of X_n, n = 1,2,... It is said that the spaces $\{X_n : n \in N\}$ have a common modulus of rotundity if

$$\inf\{\delta_n(\varepsilon) : n \in N\} > 0 \forall \epsilon : 0 < \epsilon \le 2.$$

The $\inf\{\delta_n(\varepsilon) : n \in N\} > 0$ if and only if there is one function $\delta(\epsilon) > 0$ which can be used in place of all $\delta_n(\varepsilon)$.

Definition 2.4

[7, Definition 1.5] A Fre'chet space ((F) for short) is a locally convex topological vector space. Trivial example of a Fre'chet space is a Banach space. Every Banach space (in particular, $l^p \forall 1 \le p \le \infty$) is a Fre'chet space.

Definition 2.5

[9, Definition 2.1] Let (Y,‖ . ‖) be a Banach space with basis $\{e_i : i \in I\}$ (unconditional if I is non-countable) and such that, for every finite subset J of I,

(5.0) $0 \le |\alpha j| \le \beta j, \forall j \in J \Rightarrow \| \sum_{j\in J} \alpha j e j \| \le \| \sum_{j \in J} \beta j e j \|$.

Let $\{X_i : i \in I\}$ be a family of Banach spaces. Let us consider the space

Y (Xi : i ∈ I) =‖ {‖x = (‖ ∑xi)i∈I‖∈ Π‖ i∈I‖Xi : ∑i∈I ‖xi‖ei ∈ Y }, ∈

endowed with the norm x = i∈I xi ei Y . The space Y (Xi : i I) with this norm is a Banach space.

Results and discussion

In this section we give the results. We begin with the following Lemma.

Lemma 3.1

[4, Lemma 2.2] Let (Y,‖ . ‖) be a Banach space with basis $\{e_i : i \in I\}$ satisfying the condition (5.0). Let $X_i : i \in I$ be a family of Banach spaces. We define ϕ a mapping from z = Y (X_i : i ∈ I) into Y as ϕ((x_i)i ∈ I) = ∑i∈I ‖ X_i ‖ e_i, ∀ (x_i) ∈ Z. Then ϕ has the following properties: (i) ∀z ∈ Z,‖ ϕ(Z) ‖=‖ Z ‖, and (ii) ∀Z_1,Z_2 ∈ Z,‖ ϕ(Z_1) + ϕ(Z_2) ‖≥‖ Z_1 + Z_2 ‖.

This result leads us to the following theorem.

Theorem 3.2

Let (E,‖ . ‖) be a Frechet space with basis $\{\{e_n : n \in N\}$ and such that

$$0 \le \alpha_n \le \beta_n, \forall n \in \mathbb{N} \Rightarrow \| \sum_{n=1}^{\infty} \alpha_n e_n \| \le \| \sum_{n=1}^{\infty} \beta_n e_n \|$$

Let

$$\{\varepsilon_n : \delta_E(\varepsilon) \le \delta_{E_a}(\varepsilon) \forall n \in \mathbb{N}, E \in \varepsilon_a\}$$

be a family of Frechet spaces with the property (UR) and a common modulus of rotundity.

Let

$$E(\varepsilon_1, \varepsilon_2 ...) = \{\varepsilon = (\varepsilon_n) \in \prod_{n=1}^{\infty} \varepsilon_n : \sum_{n=1}^{\infty} \| \varepsilon_n \| e_n \in E\}$$

equipped with the norm

$$\| \varepsilon \| = \| \sum_{n=1}^{\infty} \| \varepsilon_n \| e_n \|_E.$$

Then if (E,‖ . ‖) is uniformly rotund, E(ε1,ε2...) is uniformly rotund too.

Proof. Let δ : [0,2] → [0,1] be the common modulus of rotundity of $\varepsilon_n = X_n$, n ∈ N and δ_1 : [0,2] → [0,1] that of E = Y . Let us consider x = (x_n), $x^1 = (x^1_n)$ elements in E = Y ($X_1 = \varepsilon_1$,$X_2 = \varepsilon_2$...) with ‖ x ‖=‖ x^1 ‖= 1, let ε > 0 be given and suppose that ‖ $x - x^1$ ‖≥ ε. We wish to prove that ∃ $\delta_2 > 0$ such that $\| \frac{x-x^1}{2} \| \le 1-\delta_2$

This we do in two steps:

Step(1): We assume that ‖ x_n ‖=‖ (x'_n) ‖ ∀ n ∈ N.

Then ∀ n ∈ N,

$$\| x_n + (x_n)' \| \le 2(1 - \delta(\frac{\|x_n - (x'_n)\|}{\|x_n\|}))\| x_n \|,$$

since x_n and (x'_n) both lie on the sphere of radius ‖ x_n ‖ about the origin in $\varepsilon_n = X_n$.

$$(1) \|x + x'\| = \| \sum_{n=1}^{\infty} \|x_n + x'_n\| e_n \| \le$$

Hence $2 \| \sum_{n=1}^{\infty} (1 - \delta(\frac{\|x_n - x'_n\|}{\|x_n\|}))\|x_n\| e_n \|.$

Consider

$$P = \{n \in \mathbb{N} : x_n \neq 0, \frac{\|x_n - x'_n\|}{\|x_n\|} > \frac{\varepsilon}{4}\}.$$

Therefore, if

$$n \in \mathbb{N} \backslash P, \|x_n\| \ge \frac{4}{\varepsilon} \|x_n - x'_n\|.$$

Then we get

$$1 = \| \sum_{n=1}^{\infty} \|x_n\| e_n \| \ge \| \sum_{n \in \mathbb{N} \backslash P} \|x_n\| e_n \| \ge \frac{4}{\varepsilon} \| \sum_{n \in \mathbb{N} \backslash P} \|x_n - x'_n\| e_n \|$$

that is

$$\| \sum_{n \in \mathbb{N} \backslash P} \|x_n - x'_n\| e_n \| \le \frac{\varepsilon}{4}.$$

In addition,

$$\| \sum_{n \in P} \|x_n - x'_n\| e_n \| \ge \frac{3\varepsilon}{4}.$$

Hence

$$\| \sum_{n \in P} \|x_n\| e_n \| \ge \frac{1}{2} \| \sum_{n \in P} \|x_n - x'_n\| e_n \| \ge \frac{3\varepsilon}{8},$$

in view of ‖xn − x′n‖ ≤ 2‖xn‖, ∀ n ∈ N.

Now we denote

$$y = \sum_{n \in \mathbb{N} \backslash P} \|x_n\| e_n,\ y' = \sum_{n \in P} \|x_n\| e_n,\ y''(1 - 2\delta(\frac{\varepsilon}{4}))y'$$

elements in E = Y that satisfy

$$\|y + y''\| \le \|y + y'\| = 1, \|y + y' - (y + y'')\| = 2\delta(\tfrac{\varepsilon}{4})\|y'\| \ge 2\delta(\tfrac{\varepsilon}{4})\tfrac{3\varepsilon}{8} = \alpha(\varepsilon)$$

By virtue of the property (UR) for

$E=Y,\ \frac{1}{2}\|y+y'+y+y''\|\leq 1-\delta_1(\alpha(\varepsilon))$, that is

From (1) and
(1) $\|x+x'\|=\|\sum_{n=1}^{\infty}\|x_n+x_n'\|e_n\|\leq$

$2\|\sum_{n=1}^{\infty}(1-\delta(\frac{\|x_n-x_n'\|}{\|x_n\|}))\|x_n\|e_n\|$. (2),
we obtain

$\leq\|(1-\delta(\frac{\varepsilon}{4}))y'+y\|\leq 1-\delta_1(\alpha(\varepsilon))=1-\delta_0(\varepsilon)$,
where we have denoted

$$\delta_0(\varepsilon)=\delta_1(\alpha(\varepsilon))=\delta_1(\delta(\tfrac{\varepsilon}{4})\tfrac{3\varepsilon}{4}).$$

Step (2) In the general case, we suppose only that $\|x\|=\|x'\|=1$ and that $\|x+x'\|>2(1-\delta_1)$ where $0<\mu\leq 2$.
The lemma 1 shows that
$2(1-\delta_1(\mu))<\|x+x'\|\leq\|\phi(x)+\phi(x')\|$,
then
$\|\phi(x)-\phi(x')\|=\|\sum_{n=1}^{\infty}(\|x_n\|-\|x_n'\|)e_n\|<\mu$
Now let $\varepsilon_n=\pm1, n=1,2,3....$ It is worth checking that
$\|\sum_{n=1}^{\infty}\varepsilon_n(\|x_n\|-\|x_n'\|)e_n\|<\mu$.
To do this,
let
$E_a=\{n\in\mathbb{N}:\varepsilon_n=1\},\ E_a'=\{n\in\mathbb{N}:\varepsilon_n=-1\}$
We
define

$$y_n=\begin{cases}x_n, & if\ n\in E_a\\ x_n', & if\ n\in E_a', n=1,2,3,...\end{cases}$$
$$y_n'=\begin{cases}x_n', & if\ n\in E_a\\ x_n, & if\ n\in E_a', n=1,2,3,...\end{cases}$$

Then $y=(y_n), y'=(y_n')$ are elements in E = Y ($X_1=\varepsilon_1$, $X_2=\varepsilon_2$,...) such that $\|\phi(y)+\phi(y')\|=\|\phi(x)+\phi(x')\|>2(1-\delta_1(\mu))$, then $\|\phi(y)+\phi(y')\|<\mu$. We now define

$$x_n''=\begin{cases}\frac{\|x_n\|}{\|x_n'\|}, & if\ x_n'\neq 0\\ x_n, & if\ x_n'=0, n=1,2,3,...\end{cases}$$

and obtained $\|x_n''\|=\|x_n\|,\ \forall\ n\in\mathbb{N}$,
then $x''=(x_n'')\in E=Y(X_1=\varepsilon_1, X_2=\varepsilon_2,...)$ and $\|x\|=\|x\|=1$. Also
$\|x'-x''\|=\|\sum_{n=1}^{\infty}(\|x_n'-x_n''\|)e_n\|=\|\sum_{n=1}^{\infty}\varepsilon_n(\|x_n\|-\|x_n'\|)\|<\mu$, where $\varepsilon_n=\pm1$ appropriately.
It is true that
$\|x+x''\|\geq\|x+x'\|-\|x''-x'\|>2(1-\delta_1(\mu))-\mu=2(1-\delta_1(\mu)-\frac{\mu}{2})$ once $\varepsilon>0$ is fixed, we take μ such that $0<\mu<\frac{\varepsilon}{2}$ and $\delta_1(\mu)+\frac{\mu}{2}<\delta_0(\frac{\varepsilon}{2})$, where δ_0 has been defined in step (1). Then $\|x+x''\|>2(1-\delta_0(\frac{\varepsilon}{2}))$.
From step(1) we have $\|x-x''\|<\frac{\varepsilon}{2}$.
Hence
$\|x-x'\|\leq\|x-x''\|+\|x''-x'\|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$.
Therefore, given $0<\varepsilon\leq 2$, μ has been defined and hence we can find $\delta_2(\varepsilon)=\delta_1(\mu)$, such that if $x,x'\in E=Y$ ($X_1=\varepsilon_1, X_2=\varepsilon_2$ ...) with $\|x\|=\|x'\|=1$ and $\|\frac{x+x'}{2}>1-\delta_2(\varepsilon)\|$, then $\|x-x'\|<\varepsilon$.

Corollary 3.3:

Let $(Y,\|.\|)$ be a Banach space with basis $\{\{e_n:n\in N\}$ and such that $0\leq\alpha_n\leq\beta_n,\ n\in\beta\ \forall N\Rightarrow\|\sum_{n=1}^{\infty}\alpha_n e_n\|\leq\|\sum_{n=1}^{\infty}\beta_n e_n\|$ β. Let $\{X_n:n\in N\}$ be a family of Banach spaces with the property (UR) and a common modulus of rotundity. Let $Y(X_1,X_2...)=\{x=(x_n)\in\prod_{n=1}^{\infty}X_n:\sum_{n=1}^{\infty}\|x_n\|e_n\in Y\}$ equipped with the norm $\|x\|=\|\sum_{n=1}^{\infty}\|x_n\|e_n\|_Y$. Then if $(Y,\|.\|)$ is uniformly rotund, Y (X_1,X_2...) is uniformly rotund too.
Proof. Follows trivially.

Conclusions

Rotundity in Banach spaces has been studied over a period of time. Various notions of rotundity have been considered with very interesting results obtained [1-3]. Let *X* be a Banach space and a Hilbert space is a special type of Banach spaces. We have shown that uniform and most rotundity exist in Frechet spaces.

Conflicts of interest

Authors declare no conflict of interest.

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